Theory of Reproduction and Accumulation

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by

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Prepared with the collaboration of
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on the basis of lectures delivered at Warsaw University

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PREFACE TO THE FIRST AND SECOND EDITIONS

This book is, in a sense, a continuation of my *Introduction to Econometrics*. Like the *Introduction to Econometrics* it is the result of lectures delivered at the Department of Political Economy of Warsaw University.

Detailed notes of these lectures were taken by Mr. A. Banasiński who also helped me in editing this work, for which I would like to express my thanks.

The students who attended the lectures were already acquainted with the *Introduction to Econometrics* and, therefore, in my exposition I concentrated on a number of other problems.

The main subject of the book is a mathematical interpretation of the Marxist theory of reproduction and accumulation. In the first two chapters an attempt is made to interpret this theory in a rigorously mathematical way; then multi-branch schemes of reproduction are discussed, followed by an analysis of inter-branch flows.

I then analyse the influence of investment on economic growth and, in this connection, design a model of economic growth in which I try to explain both the causes of business cycles in a capitalist economy, and the reasons why business cycles do not exist in a socialist economy. In contradistinction to most contemporary western writings on mathematical economics which explain the phenomenon of business cycles without accounting for the phenomenon of economic growth (trend), the model described in this book—in accordance with the Marxist theory of capitalist reproduction—introduces both the trend and the business cycle as peculiarities of the development of a capitalist economy. I took advantage of Michał Kalecki's ideas which constituted a basis of his theory of business cycles. The main ideas of this chapter were published in *Ekonomista*, No. 3, 1959 in the paper: A Model of Economic Growth (Model wzrostu gospodarczego). In the second edition of this book the mathematical interpretation has been improved and expanded.

The last chapter deals with the theory of depreciation and replacement. This problem is of special importance to a socialist economy. The application of the tools of mathematical demography make it possible to plan more efficiently for the replacement of production facilities. In this context, I also discuss an interesting problem of re-investment cycles in a socialist economy and show that whilst trends toward such cycles may develop in consequence of an initial cumulation of investments, they disappear quickly against the background of general growth of a socialist economy.

The problems discussed in this book are interpreted in terms of mathematical economics. The results lend themselves, however, to an economic application in practice.

The book contains two appendices. One describes a model designed by the author for the purpose of solving the problem of planning inter-branch flows and planning for economic growth. If, for instance, the existing productive capacity and the planned rate of growth of the particular sectors of the economy, as well as the planned level and structure of consumption are given, this model enables us to determine the investments required for this purpose, their material structure and allocation to different sectors of the national economy. From this model we can also read off the total production of each sector of the national economy resulting from the objectives of the plan.

This model can be constructed both in a hydraulic and an electrical version. A simplified electric model was constructed by the staff of the Chair of Automation at the Metallurgy and Mining Academy of Cracow, under the direction of Professor Henryk Górecki in co-operation with Professor Bolesław Kłapkowski, Chairman of the Chair of Political Economy of the Academy. The Warsaw University model is now at the Department of Political Economy and is being used in experiments.

The second appendix provides general information on differential and difference equations. It will help the reader in understanding chapters 4 and 5 of this book. For this reason it should be read before these two chapters. Appendix 2 was written by Mr. A. Banasiński.

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Chapter 1

GENERAL THEORY OF REPRODUCTION

In this chapter we shall deal with the general theory of reproduction and accumulation, with the influence of accumulation on production growth and with a number of particular problems relating to depreciation and replacement.

The problem of reproduction and accumulation appears in every socio-economic system.

In every system there is production which consists in making material objects for the purpose of satisfying human needs. The activity that produces these goods is human labour. However, the objects used for the satisfaction of needs are not made of nothing. They are made of materials provided by Nature.

In the process of production we have, then, the following elements: materials provided by Nature, called *objects of labour*, and *human labour* which transforms these materials. Moreover, even in a most backward society human labour is not performed with bare hands but with the help of material objects called *means of labour*. Some of them, the most specialized ones, are called *tools of labour*; others, such as factory buildings, which are not of so specialized a nature, are called *auxiliary equipment*.

The process of production consists in transforming objects of labour by human labour, with the help of means of labour. In other words, the process of production is a process performed on objects of labour with the help of means of labour.

Objects and means of labour are usually called *means of production*; therefore, the process of production can also be defined as the making of certain objects called *products* by the application of labour to certain means of production.

This nature of the process of production is universal and independent of any social system. Only in most primitive societies may there not exist means of labour (such a society is called a "society of gatherers").

In the process of production, objects of labour and means of labour are used up in different ways. Objects of labour are usually completely transformed into products (such objects of labour are called *materials*), and means of labour are gradually worn out during the process of production, some faster, some more slowly. Some means of production are used up completely during one production period (cycle), others may be used in production over a long period of time and their period of wear and tear may last for several years (e.g. buildings). The former, i.e. the ones which are used up in one production cycle are called *working capital means*, and the others—*fixed capital means*.

If a process of production is of a constant nature, the used up means of production must be replaced by new ones. Thus, in the course of production it is necessary to replace the used up means of production. This process of replacement of means of production is called *reproduction*.

Some means of production, namely, working capital means, are replaced immediately, before the start of a new production cycle, while fixed capital means of production are replaced gradually, as they wear out.

The theory of reproduction deals with the problem of the replacement of used up means of production and endeavours to ascertain the consequences which may ensue if means of production were not replaced in time or if they were not fully replaced.

It is obvious that if the amount of the means of production decreases normally, at a given level of technology, this results in a decline in the amount of the product. The level of technology prevailing in production can be determined by the *magnitude of productivity* by which we understand the amount of product obtained per unit of labour input and used up means of production. The measuring of productivity, thus defined, is a separate and complex problem.

Productivity may increase owing to *technical progress*. Technical progress, understood in the broadest sense of this word, that is, interpreted also as improvements in the organization of production, makes it possible to maintain the existing level of production without full replacement of used up means of production. For the time being, in our further considerations, we shall disregard technical progress, i.e. we shall assume that the conditions of production do not change. Under these circumstances, to maintain a steady level of production from one period to the next, we must replace the used up means of production and also have at our disposal, during each production period, the same number of *the labour force*, i.e. the same amount of potential capacity to perform the work.

We shall analyse more closely the role of labour in the production process. Labour may be looked upon in a similar way as means of production, and we can speak of its reproduction similarly as we speak of the reproduction of means of production. However, the matter is not so simple and it can easily be seen that the role of labour in the production process depends upon the social system. Indeed, in a slave system, there is really no difference between the reproduction of means of production and the reproduction of slave labour. In ancient Rome, the process of replacement of slaves employed in production was the same as the process of replacement of cattle.

In other social systems the situation is different, and the reproduction of labour takes place in a different way from the reproduction of means of production. The fact is, however, that under given technical conditions, to maintain the process of production at a steady level, the supply of labour must also be maintained at a steady level.

The production process in which the means of production are fully replaced and in which the amount of labour is constant (and, moreover, as we have assumed above, there is no technical progress) is called *the process of simple reproduction.* Under conditions of simple reproduction, the national economy does not expand; this means that it is stationary.

If during the process of production the means of production are not fully replaced, or if the amount of labour employed decreases, we are faced with *contracted reproduction.* If, however, in the process of reproduction the means of production are not only fully replaced but their quantity is increased from one period to another, and the amount of labour also increases sufficiently to use the additional means of production, we speak of *expanded reproduction.*

As we know, the process of economic development consists in expanding reproduction and in increasing productivity which, for the time being, we shall not discuss.

The process of increasing the amount of means of production together with a simultaneous increase in employment is called *accumulation*. Thus, in case of expanded reproduction there must inevitably be accumulation. But in the process of contracted reproduction *decumulation* (or negative accumulation) occurs; it consists in decreasing the amount of means of production as well as of employment. Finally, in the case of simple reproduction accumulation equals zero.

By studying various production processes we can discover certain laws which are called the *technical balance-sheet laws of production*.[1] We shall deal with them presently. Their very name suggests their nature and origin.

First of all, there exist in the process of production certain relations of a technical nature: in order to produce commodities it is necessary to use defined quantities of means of production and of labour. For instance, to produce 1 ton of steel certain quantities of iron ore and coal are required as well as certain fixed capital means (blast furnaces, buildings, transportation means, etc.), and a certain amount of labour. The amount of means of production and of labour required to produce a unit of a given product is expressed by the *technical coefficients* or *technical norms*, determined on the basis of the technological process used.

In addition to the relationships of a technical nature described above there are others resulting from the availability in every production process of certain supplies of material objects (means of production) and of labour. These factors of production are always scarce, and if more of them are used up for one purpose, there are fewer left for other purposes. For example, in the process of producing steel, we cannot use up more coal than we obtain from current extraction, imports and available stocks. These relations are called *balance-sheet relations* because they are usually expressed in the form of a balance-sheet of means of production and of labour.

Technical balance-sheet laws are, as a rule, independent of any social system. They are of a historical nature, similarly as is the

[1] See Oskar Lange, *Political Economy*, Pergamon Press, London– PWN, Warsaw, 1959, p. 58 ff.

development of the productive resources of society. Technical balance-sheet laws depend upon productive resources and do not depend directly upon social relations.

Different configurations of social relations affect the production process in two ways: firstly, they shape it in a specific way and, secondly, as changes occur in the social relations in which the production process takes place, other relations connected with the social relations are added to the basic technical balance-sheet laws. For instance, under capitalist conditions, the nature and magnitude of reproduction depend upon expected profits. Expanded reproduction takes place, then, primarily, in those branches of production which are more profitable. The branch of production which yields no profit gradually shrinks or disappears.

In a socialist economy the factors of production which determine the magnitude of reproduction are completely different. Reproduction in a socialist economy depends upon the plan based on other premises than incentives affecting the development of a capitalist economy.

The starting point for our further considerations will be the analysis of the basic technical balance-sheet laws of production disregarding, for the time being, the specific relationships resulting from prevailing social relations. In our analysis we shall use as an example an economy with modern productive resources prevailing both under conditions of a capitalist and a socialist economy. After having become acquainted with the laws of reproduction at the most advanced stage of development of productive resources, it will be easy to check whether these laws are applicable to other, earlier social structures. We assume here that one of the ways of understanding the functioning of primitive economic organisms is by acquiring a knowledge of the structure and functions of more advanced economic organisms, as the key to understanding the anatomy of an ape is the knowledge of the anatomy of man.[2]

The history of modern theories of production goes back to the Physiocrats who were the first to attempt a study of the whole production process. The further development of the theory of

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[2] Compare the opinion on this subject contained in K. Marx's essay, An Introduction to the Critique of Political Economy in *Przyczynek do krytyki ekonomii politycznej* (*A Contribution to the Critique of Political Economy* in Polish), Warsaw, 1953, p. 252.

reproduction is presented in Chapter 3, Volume 2 of *Capital*. Later works in this field virtually consist in rendering Marx's analysis more concrete.

Marx's approach to the analysis of reproduction takes the form of schemes which can also be presented as formulae. Let us assume that in the process of reproduction, within a unit of time, say, one year, the used up means of production, or the input of the means of production, is c and the amount of work performed during this period is $v+m$. By using up the means of production in the amount c and the labour input in the amount $v+m$, the amount of product obtained is X.

The first problem arising from this assumption is to determine the mode of measuring quantities c and $v+m$ and the quantity of the product obtained. The quantities of the means of production could be measured in physical units and then c would consist of a whole list of various means of production (buildings, tools, raw materials, etc.) used up in manufacturing product X. The amount of work could be measured in, say, man-hours. But we should distinguish here between the specific work of various kinds of workers, as $v+m$ would represent work performed by workers of different skills and qualifications.

We must simplify the problem by expressing all quantities appearing in the production process in monetary units, e.g. in zlotys, or in units of value, e.g. in man-hours of average social labour.

It is possible to express quantities c and $v+m$ in monetary units instead of physical units because this procedure does not affect at all the essence of technical balance-sheet laws studied by us. Indeed, monetary units represent certain definite amounts of social labour and, moreover, by expressing, say, $v+m$ in monetary units we "weight" the work of various kinds of workers according, for instance, to the rule that 1 hour of work of a skilled worker equals 3 hours of work of an unskilled worker. In a capitalist economy this "weighting" is reflected on the labour market by the level of remuneration for the job. In a socialist economy "weighting" is based on properly arranged wage rates.

It should be noted, however, that assumptions of this kind limit the analysis of technical balance-sheet laws to contemporary capitalist and socialist economies. The laws arrived at on the basis of such assumptions cannot be applied directly to, say, a feudal economy. This problem would require a separate study.

By c we shall understand, then, the value of the used up means of production expressed in zlotys, and by $v+m$ we shall denote the value of labour input. Thus, we can add together the outlays on different means of production and labour input.

With these assumptions we obtain Marx's well-known formula: $P = c+(v+m)$ which means that the total input P required to obtain the given amount of product X equals the sum of c, i.e. *the value of the used up means of production* and $(v+m)$, i.e. the labour input which, in turn, is the sum of the components v, i.e. *the cost of labour*, and m, i.e. *the value of the surplus product*. The value of the surplus product m is not paid out in the process of production and constitutes a surplus of total outlays P over the value of the used up means of production c and the cost of labour employed v. The total outlay $P = c+v+m$ determines the value of the product, X.

As to formula $P = c+(v+m)$, a few comments on the terminology used are necessary.

Since the used up means of production c are also the effect of labour and represent a certain amount of labour performed earlier, to distinguish from this component the amount of labour used up now, we call the component $(v+m)$ *the input of living labour*.

Marx calls component c—*constant capital*, and v—*variable capital*. In Soviet economic literature, component m is often called *the product for society*.[3] However, for the sake of consistency in terminology, when discussing the process of production both under capitalist and socialist conditions, we shall call component m—the value of the surplus product.

In Marx's schemes component v is interpreted in the same way as component c, i.e. as the cost of reproduction of labour. The same interpretation of v, under conditions of a socialist economy, would not hold because in a socialist economy the whole component of the product corresponding to labour input, i.e. $(v+m)$, even if it is partly earmarked for, say, accumulation, is actually used up for the needs of the society. The part of labour input,

[3] This term was introduced by Józef Stalin in: *Ekonomiczne problemy socjalizmu w ZSRR* (*Economic Problems of Socialism in the U.S.S.R.*, in Polish), Warsaw, 1953, pp. 60–61.

denoted by v is, as we know, direct remuneration for labour. While the level of this remuneration cannot go below a certain limit, corresponding to the cost of replacing used up labour, it may be higher. A socialist economy strives to raise this level gradually.

Let us also comment upon the units of measurement corresponding to the quantities P, c, v and m. All these quantities are expressed in units of social labour or in monetary units per unit of time usually "per year". Using the method of determining the dimension of the units of particular economic quantities, analogous to the method used in physics,[4] we state that the dimension of the quantities P, c, v and m should be written symbolically as W/T, or WT^{-1} which means that these quantities are expressed in units of value W (e.g. in units of social labour or in monetary units) per unit of time T (e.g. per year). To know the proper dimension of the economic quantities studied is of great importance because it enables us to avoid misunderstandings.[5]

The use of the terminology introduced by Marx, without paying attention to the unit of measurement of the quantities studied, may lead to serious errors. For instance, Marx assumed in his considerations that all constant capital c is used up during one production cycle of one year's duration. In this case the

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[4] Let us remember that units of all physical quantities can be expressed by the units of the system (cm, g, sec), i.e. centimetre, gram, second, or the dimensions of the system (L, M, T), i.e. length, mass, time. The units in which a physical quantity is expressed constitute its dimension. E.g. the dimension of a unit of velocity is written symbolically as $\frac{L}{T} = LT^{-1}$; the dimension of a unit of acceleration: LT^{-2}; the dimension of a unit of force: MLT^{-2}; the dimension of a unit of labour: ML^2T^{-2}, etc.

[5] W. S. Jevons was the first to use consistently in economics the method of determining the dimension of the quantities studied. In his book, *The Theory of Political Economy*, published in 1871, he devoted to this problem a separate chapter. Jevons made certain errors in determining the dimension of economic quantities. These errors were corrected by P. H. Wicksteed. See P. H. Wicksteed, *The Common Sense of Political Economy*, vol. 2, London, 1946, Supplement: "Dimensions of Economic Quantities". This supplement is a reprint of the paper published in *Palgrave's Dictionary of Political Economy*. See also S. C. Evans, *Mathematical Introduction to Economics*, New York, London, 1930, Chapter 2.

value of c has the dimension W, i.e., it is expressed simply in units of value. If, however, constant capital, as usually happens, does not wear out in the course of one production period, then the dimension of c is WT^{-1}, and thus c is expressed in units of value per unit of time (year).

A clear distinction must, therefore, be made between quantities with dimension W (e.g. the total value of stocks of means of production used in production) and quantities with dimension WT^{-1}, which is the dimension of, say, quantity c (or v) denoting the value of means of production (or labour) used up in production in the course of one year.

The quantities expressed in dimension W will be called *stocks* and the quantities expressed in dimension WT^{-1}—*flows*.[6] Economists who are not accustomed to thinking in precise mathematical terms do not always realize the dimension of the quantity about which they speak. Worth remembering is a saying of one of the prominent economists (M. Kalecki) who once jokingly remarked that "economics is a science in which the notions of stocks and flows are always confused and, therefore, errors are committed".

Let us illustrate the quantities discussed by using some relevant statistical examples. On the basis of data from the *1960 Statistical Yearbook* (pp. 66–67) and from the publication of the Central Statistical Office, *Poland's National Income 1955–1960* (*Dochód narodowy Polski 1955–1960*) the following quantities of c, v, m and P can be obtained for the whole Polish economy in 1957, 1958 and 1959.

Poland	In millions of zlotys per year (at current prices)
1957	$392878.3\,c + 179617.0\,v + 121825.0\,m = 694320.3\,P$
1958	$427226.9\,c + 190382.6\,v + 130951.1\,m = 748560.6\,P$
1959	$469214.0\,c + 200283.4\,v + 145543.2\,m = 815040.6\,P$

The method by which we have obtained the figures shown in the table above requires an explanation. Marx's schemes, as we

[6] Analogously, the water in a container is called the stock of water and is measured in, say, litres, and the water flowing out of the container is called a flow and is measured in, say, litres per second.

know, apply to a pure capitalist economy, i.e. they comprise the workers employed in production and the capitalists who derive income from a surplus product. According to analogous rules, schemes can be prepared for a pure socialist economy. In our conditions certain complications arise, however, due to the existence of a small private sector. It is, therefore, necessary to add to component v a part of income from private peasant farming and from the non-socialized sector outside agriculture.

Thus, included in v are:

(1) Both the regular employment and the contract work wage funds for those employed in material production;

(2) Net production from peasant farming less land tax, and income from additional sources earned by the rural population;

(3) Net income from the non-socialized sector outside agriculture, i.e. net production less taxes, and additional income of the urban population (collection of usable waste, building on one's own account).

Included in component m for socialized enterprises are: taxes, budget differences, profits (or losses), social insurance, non-material costs outside regular employment and the contract work wage funds. Also included in m are taxes from the non-socialized sector.

The estimate obtained is a little too high for v and too low for m, since net investments from means coming from the non-socialized sector are here included in v, while they should be included in m. For the time being, there are no sufficiently detailed data available to separate this item and, therefore, no attempt was made to introduce this correction. In spite of this inaccuracy, the figures quoted give some idea concerning the magnitudes of the particular components of Poland's social product.

Similar statistical information on the value of the gross national product P and of its components c, v and m for Great Britain in 1950 was obtained by V. S. Nemchinov,[7] member of the Academy

[7] "Nekotorye voprosy ispolzovanya balansovogo metoda v statistikye vzaimnosvyazanykh dynamicheskikh ekonomicheskikh system", *Doklady sovetskikh uchonykh na XXXI sesyu Myezhdunarodnogo Statisticheskogo Instituta*, Moscow, 1958, Reprinted in *Uchenye zapiski po statistike*, vol. 5, Moscow, 1959.

of Sciences of the USSR, on the basis of inter-branch flows prepared for the economy of Great Britain.[8]

The results obtained, after certain adjustments made by us,[9] are as follows:

Great Britain	In millions of pounds for the year
1950	$7677\,c+6225\,v+7003\,m = 20905\,P$

On the basis of statistical data of this kind it is possible to determine certain relations existing in the national economy. These relations may concern either the structure of the gross national product by value or the outlays made during the process of production. Let us begin by discussing the relations of the first type, i.e. certain *structural relations*.

First of all, we single out part $Y = v+m$ of the gross national product which we shall call *the total value added in a year*, or *the national (social) income*.

The term "value added" is used because the society, by its productive labour, replaces not only the used up means of production but also adds a certain amount over and above component c. This value added $Y = v+m$ corresponds to the input of living labour, expressed in monetary units. The first interesting structural relation is the ratio of the value added Y to the value of the gross national product P, i.e. $\frac{Y}{P} = \frac{v+m}{c+v+m}$, which we shall call *the income–product ratio*.

The reciprocal of the income–product ratio, i.e. the ratio $\frac{P}{Y} = \frac{c+v+m}{v+m}$ we shall call *the efficiency of living labour*.[10]

[8] Examples of tables of this kind are given in the Appendix to the book by Oskar Lange, *Introduction to Econometrics*, Pergamon Press, London–PWN, Warsaw, 1962. See also *1960 Statistical Yearbook* (In Polish), pp. 70–73.

[9] Nemchinov gives as the value of component c the amount of 4868 million, because he does not include in component c the value of imports of means of production amounting to 2809 million.

[10] The ratio P/Y is called productivity when the quantities P and Y are expressed in physical units.

The income–product ratio Y/P calculated on the basis of the statistical data given above is:

for Poland in 1957—0.43
,, ,, ,, 1958—0.43
,, ,, ,, 1959—0.42
for Great Britain
in 1950—0.63.[11]

Analogous data pertaining to the national economy in other countries show that, for instance, the income–product ratio in the United States in 1947 was 0.45, and 0.56 in the Soviet Union for the fiscal year 1923/24, according to Nemchinov's calculations.

It follows from these data and from other statistical studies that the ratio of national income to gross national product is approximately 0.5, or slightly less. Thus, usually more than one half of the gross national product is used for the replacement of the values used up during the production of means of production, and slightly less than a half of the gross national product constitutes the value added or the national income.

The efficiency of living labour is slightly less than 2, which means that each unit of living labour produces a little more than 2 units of the gross national product. For instance, 1000 zlotys worth of living labour produces a little more than 2000 zlotys worth of the gross national product.

Diagramatically, the process of reproduction may be presented as follows:

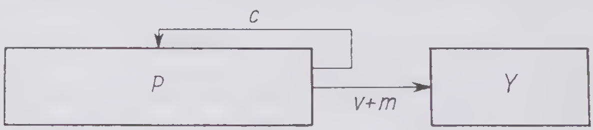

Fig. 1.

This process can be explained most figuratively by the example of agricultural production. Farmers harvest grain and use part of the crop for sowing in the next production year; the remainder constitutes the farmers' income.

[11] If the value of imported means of production were not included in component c, the ratio Y/P would be 0.73.

Very interesting are the fluctuations in the income–product ratio in different years and in particular branches of production.

First of all it should be stated that in capitalist countries there appear distinct fluctuations in the total income–product ratio; these are related to the course of the business cycle.

Studying the magnitude of the income–product ratio in various branches of production, we find (as might be expected) that the sectors of the national economy better equipped in means of production have usually a lower income–product ratio. This is explained by the fact that with an increasing supply of means of production in a given sector of the economy, the share of the replacement of means of production increases and simultaneously the share of the component $Y = v+m$ in the gross national product decreases. In consequence, the ratio Y/P decreases.

In agriculture, for instance, the replacement of means of production is relatively smaller (in relation to the gross national product) and, therefore, in this sector of the economy, value added constitutes a major part of the product. In industry the situation is reversed, namely, a relatively large part of the gross national product is used for the replacement of used up means of production.

Below are given some data on the income–product ratio in various countries and in different branches of the economy. It follows from the data obtained from the *1960 Statistical Yearbook* (pp. 66–67) that the Y/P ratio in Poland in some sectors of the national economy was (in percentages):[12]

	1957	1958	1959
Manufacturing and handicraft	36.1	36.0	35.6
Construction	50.7	51.9	51.0
Agriculture	44.5	44.5	44.1

[12] The reciprocal of the income–product ratio is the efficiency of living labour. As can be seen from the figures quoted, this efficiency was the lowest in manufacturing, then in agriculture and in the third place—in construction.

The income–product ratio for agriculture and for agricultural industries in Great Britain was 44.8 per cent in 1955, and for the same sectors of the economy in the United States it was 55.2 per cent, which is probably due to more extensive farming in the United States than in Great Britain.

Analogous data for the coal and power industry were 80.2 per cent in Great Britain and 55.2 per cent in the United States, and for the metalurgical industry—64.6 per cent in Great Britain and 40.4 per cent in the United States.

Let us consider now two further structural relations:

(1) *The organic composition of inputs*, k,[13] which is the ratio of the values of used up means of production c to the value of labour input v,[14] i.e. $k = c/v$,

(2) *The rate of surplus product* s is defined as the ratio of the value of surplus product to labour input remunerated in the form of wages; thus $s = m/v$; under capitalist conditions, we call this ratio—in accordance with Marx—*the rate of surplus value.*

Statistical values of these ratios are:[15]

for Poland	in 1957	$k = 2.19$	$s = 0.68$
,, ,,	,, 1958	$k = 2.24$	$s = 0.69$
,, ,,	,, 1959	$k = 2.34$	$s = 0.73$
,, Great Britain	,, 1950	$k = 1.23$	$s = 1.12$

According to the information given by Strumilin, the rate of surplus value in Russia before the revolution was about 1 and in the Soviet Union (after World War 2) the rate of surplus product was $s = 0.74$.[16]

According to J. M. Gillman, the rate of surplus value in manufacturing in the United States increased from 1.11 in 1920 to

[13] The notion of the organic composition of inputs should not be confused with the notion of *the organic composition of capital*, introduced by Marx, who understood by it the ratio of constant capital engaged in means of production K, to variable capital Z.

[14] The value of labour is the labour input remunerated in the form of wages.

[15] Marx in his examples assumed: $s = 1$ and $k = 4$ (e.g. $P = 4000\,c + 1000\,v + 1000\,m$).

[16] *Nekotorye voprosy ispolzovanya balansovogo metoda v statistike vzaimnosvyazanykh dynamicheskikh ekonomicheskikh system*, p. 11.

1.64 in 1929. Then, due to the economic depression, the rate of surplus value declined to 1.36 in 1935 and afterwards it began to climb again to reach 1.49 in 1939. The organic composition of inputs in manufacturing fluctuated during this period between 3.4 and 4.0.[17]

The high rate of surplus value in the United States is due to the fact that the surplus product contains the capitalists' income, part of which is invested, and part is used to cover non-productive expenditure, primarily for all kinds of services. In the United States services of this kind (e.g. advertising expenditure) are more developed than in the European countries.

On the basis of numerous statistical studies certain regularities in this field have been established. It became apparent that, with the development of society, the share of expenditure for services increases, i.e. an ever growing portion of national income is earmarked for the satisfaction of needs by services. First, food requirements are satisfied, then comes clothing, household furniture and equipment etc. and finally, with further economic development, relatively higher expenditure goes to satisfying needs by services. This results in an increase in s. However, transition from the capitalist to the socialist system causes a decline in s because the part of the social product consumed by the capitalist and by related social strata disappears.

With regard to the organic composition of inputs, the data given above show that in Poland this composition gradually grows. It is difficult, however, to explain why the computed organic composition of inputs in Great Britain is lower than in Poland.

The organic composition of inputs $k = c/v$ would be identical with Marx's notion of the organic composition of capital defined as the ratio of constant capital K, engaged in production, to variable capital Z, only if the turnover period of constant capital and the turnover period of variable capital were the same and, as Marx assumed, if they amounted to one year.

In reality, however, this is not so. Fixed capital means (buildings, machinery, equipment, etc.) can be used in production, on the average, for 20–30 years, and working capital means (raw materials) are, as a rule, used up faster than within one year.

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[17] Joseph M. Gillman, *The Falling Rate of Profit*, New York, 1958, p. 40.

Sometimes they are used up in several weeks and, thus have a "turnover" of several, or more, times within one year.

For this reason, Marx introduced the concept of *the turnover period of constant capital*[18] which denotes the average lifespan of means of production used in the process of production; this refers to the average economic lifespan and not necessarily to the physical wear and tear. This period is denoted by τ.

The reciprocal of the period of capital turnover τ is called the *rate of replacement*[19] and is denoted by μ. Hence $\mu = 1/\tau$. If, for instance, the period of capital turnover is $\tau = 10$ years, then the rate of replacement is $\mu = 1/10$.

If we denote by K the constant capital engaged in production, then the following relationship holds:

$$c = K\mu = K\frac{1}{\tau} \qquad \text{or} \qquad K = c\tau.$$

The situation is similar for variable capital Z, which is not the same as the annual wage fund v. These quantities would be identical only if the period of paying wages for work was 1 year.

When the period of variable capital turnover—let us denote it by θ—is, say, 1 month, i.e. $\theta = \frac{1}{12}$, variable capital amounts only to $\frac{1}{12}$ of the annual wage fund v multiplied by the period of variable capital turnover, i.e.: $Z = v\theta$.

Therefore, the organic composition of capital, according to Marx's definition, equals: $\frac{K}{Z} = \frac{c\tau}{v\theta} = k\,\frac{\tau}{\theta}$, i.e. the product of the organic composition of inputs by the ratio of the period of constant capital turnover to the period of variable capital turnover.

It follows that only in some special case, when $\tau = \theta$, does the organic composition of capital equal the organic composition of inputs, i.e. $\frac{K}{Z} = \frac{c}{v}$.

[18] See *Capital*, vol. 2, pp. 159 and ff.

[19] The concept of the rate of replacement should not be confused with the *rate of depreciation*. By depreciation we understand the writing off of a certain amount of money for replacement, but this sum need not equal the actual wear and tear that occurred within a given period. The determination of the rate of depreciation is related to the problem of financing reproduction.

Let us now check the dimensions of the quantities discussed above. The dimension of constant capital K, as stock, is W, and the dimension of the value of means of production used up within one year, c, is WT^{-1}. The dimension of the period of constant and variable capital turnover, i.e. of τ and θ, is T.

Since $K = c\tau$, the dimension of the product $c\tau$ must correspond to the dimension of quantity K which, indeed, is the case because after multiplying the units of the right-hand side of the formula $K = c\tau$ we obtain:

$$WT^{-1}T = W,$$

i.e. the dimension that appears on the left-hand side of this formula. The situation is similar for formula $Z = v\theta$.

Let us now introduce a further coefficient called the *rate of profit* p. This is the ratio of the value of surplus product to the sum of constant and variable capital:

$$p = \frac{m}{K+Z}.$$

This formula can be transformed as follows:

$$p = \frac{m}{K+Z} = \frac{m}{c\tau+v\theta} = \frac{\frac{m}{v}}{\frac{c\tau}{v}+\theta} = \frac{s}{k\tau+\theta} = \frac{s}{\left(k\frac{\tau}{\theta}+1\right)\theta},$$

and because of the relationship previously obtained:

$$\frac{K}{Z} = k\frac{\tau}{\theta},$$

we obtain the following formula:

$$p = \frac{s}{\left(\frac{K}{Z}+1\right)\theta}.$$

This formula determines the dependence of the rate of profit on the organic composition of capital $\frac{K}{Z}$, the rate of surplus product s and the period of variable capital turnover θ.

In a similar way, we can determine the rate of profit $\bar{p}$ for constant capital K:

$$\bar{p} = \frac{m}{K} = \frac{m}{c\tau} = \frac{\frac{m}{v}}{\frac{c}{v}\tau} = \frac{s}{k\tau}.$$

It follows that the rate of profit from constant capital equals the rate of surplus product s divided by the product of the organic composition of inputs and the period of constant capital turnover τ.

The latter formula can be transformed in a similar way to before, using the relationship: $k = \frac{K}{Z} \cdot \frac{\theta}{\tau}$. We obtain:

$$\bar{p} = \frac{s}{k\tau} = \frac{s}{\frac{K}{Z}\theta}.$$

From the formulae for p and $\bar{p}$ it follows that at a given rate of surplus value s and a given period of variable capital turnover θ, the rate of profit decreases when the organic composition of capital increases. This is Marx's well-known theorem on the tendency of the rate of profit in a capitalist economy to decline in consequence of a steady growth of the organic composition of capital.

However, a certain complication arises here. From the formulae:

$$p = \frac{s}{\left(\frac{K}{Z}+1\right)\theta} \quad \text{and} \quad \bar{p} = \frac{s}{\frac{K}{Z}\theta}$$

it follows that if the period of variable capital turnover could be reduced, then the rate of profit would increase even though the organic composition of capital remained the same or even increased.

This fact can easily be explained. For a shortening of the period of variable capital turnover means that in the process of production, with the same annual wage fund v, a smaller amount of variable capital $Z = v\theta$ can be used.

It is interesting to note the dimension of a unit rate of profit $p = \frac{m}{K+Z}$. Since the dimension of the numerator of this expression is WT^{-1} and of the denominator W, the dimension of the rate of profit p is $\frac{WT^{-1}}{W} = T^{-1} = \frac{1}{T}$. This is the same dimension as that of the rate of interest or of the replacement coefficient. If, for instance, $K+Z = 1000$ milliard zlotys and the value of surplus product $m = 100$ milliard zlotys per year, then the rate of profit equals 10 per cent "per annum". In this case the rate of profit, say, "per 1/2 year" would be 5 per cent and "per 2 years"—20 per cent, etc. All ratios of flows to stocks have the dimension T^{-1}.

Let us note that, on the contrary, the ratio of the stock to the flow, e.g. the period of turnover $\tau = \frac{K}{c}$, has dimension T, since $\frac{W}{WT^{-1}} = T$.

In order to examine further structural relations let us transform the formula for the gross national product $P = c+v+m$ as follows:

$$P = \left(\frac{c}{v}+\frac{v}{v}+\frac{m}{v}\right)v = (k+1+s)\,v.$$

It follows that when the organic composition of inputs k and the rate of surplus value s are constant, the gross national product is proportional to the input of labour v. The coefficient of proportionality in this relationship is the expression: $k+1+s = \frac{P}{v}$, which determines the size of the gross national product per unit labour v.

The formula for the gross national product: $P = (k+1+s)v$, applied to the statistical data pertaining to Poland, quoted above, is as follows (in zlotys, at current prices):

in 1957	$694320.3 = (2.19+1+0.68) \cdot 179617.0$
in 1958	$748560.6 = (2.24+1+0.69) \cdot 190382.6$
in 1959	$815040.6 = (2.34+1+0.73) \cdot 200283.4$

It will be helpful to use in our further analysis *the input coefficients (parameters)* which play an important part in present day input–output analyses of production. They determine the amount of input (means of production or labour) necessary to produce a unit product and, thus represent the share of means of production or labour in the value of the product.

The input coefficient of means of production, or the input of means of production per unit product is:

$$a_c = \frac{c}{P}.$$

Similarly, the labour input coefficient per unit product is

$$a_v = \frac{v}{P}.$$

We shall also introduce in our considerations the coefficient $a_m = \frac{m}{P}$ which determines the proportion of the surplus product to the gross national product, i.e. the amount of the surplus product per unit gross national product.

From these definitions it follows directly that $a_c + a_v + a_m = 1$, and that each of these coefficients is less than one. It is obvious that all the coefficients are positive.

In the table below are given coefficients a_c, a_v and a_m, determined on the basis of statistical data for Poland for the years 1957, 1958 and 1959 and for Great Britain for 1950:

Coefficients	Poland			Great Britain 1950
	1957	1958	1959	
a_c	0.566	0.571	0.576	0.37
a_v	0.259	0.254	0.246	0.30
a_m	0.175	0.175	0.178	0.33

It follows from this table that in 1959, for instance, in Poland the share of the input of means of production in the gross national product was close to 58 per cent and the share of outlays in wages was over 24 per cent. The surplus product constituted about 18 per cent of the gross national product.

Let us note that the income–product ratio (the share of the national income in the gross national product) $\frac{Y}{P}$ can be determined by the coefficients a_v and a_m. Indeed

$$\frac{Y}{P} = \frac{v+m}{P} = a_v + a_m.$$

The income–product ratio for Poland for 1959 was about 0.42 which is consistent with formula

$$\frac{Y}{P} = a_v + a_m = 0.246 + 0.178 = 0.424.$$

Also, the organic composition of inputs k and the rate of surplus product s can be conveniently expressed by the coefficients a_c, a_v and a_m. We have

$$k = \frac{c}{v} = \frac{a_c P}{a_v P} = \frac{a_c}{a_v},$$

$$s = \frac{m}{v} = \frac{a_m P}{a_v P} = \frac{a_m}{a_v}.$$

CHAPTER 2

EQUILIBRIUM CONDITIONS OF THE PROCESS OF REPRODUCTION

WE shall now analyse the conditions of the process of reproduction, assuming that the national economy is divided into a number of branches. We shall begin with an analysis of simple reproduction in which the gross national product P, consists of two divisions (departments): one containing the products which serve as means of production P_1 and the other the products which serve as means of consumption P_2. Thus: $P = P_1 + P_2$.

Dividing into components each of the two parts of the gross national product, in accordance with Marx's schemes, we obtain the following table:

$$c_1 + \boxed{v_1 + m_1} = P_1$$

$$\boxed{c_2} + v_2 + m_2 = P_2$$

$$\overline{c + v + m = P}.$$

As we know, simple reproduction occurs when the gross national product contains exactly the amount of means of production needed to replace the used up means of production. As can be seen from the table, the amount of means of production used up in the process of production in both divisions is $c_1 + c_2$ and the amount produced (in Division 1) is $c_1 + v_1 + m_1$. Since $c_1 + c_2 = c_1 + v_1 + m_1$, then

$$c_2 = v_1 + m_1. \qquad (1)$$

This is the well-known *condition of equilibrium of inter-branch flows*, discovered by Marx, which must be satisfied in the case of simple reproduction.

This condition can also be obtained in a different way. In the process of simple reproduction the whole national income $v+m$ is consumed and therefore:

$$v+m = c_2+v_2+m_2,$$

or

$$v_1+v_2+m_1+m_2 = c_2+v_2+m_2.$$

Hence

$$v_1+m_1 = c_2.$$

The scheme of simple reproduction can also be presented as an input–output table:

$$\begin{array}{cc|c} c_1 & c_2 & c \\ v_1 & v_2 & v \\ m_1 & m_2 & m \\ \hline P_1 & P_2 & \end{array}.$$

The rows of this table show how the results of production are distributed between Division 1 and Division 2. Similarly, the columns show how the production inputs are distributed.

It is evident that to maintain equilibrium in inter-branch flows the following equilibrium conditions must be satisfied:

$$c = P_1 \quad \text{or} \quad v+m=P_2.$$

This leads to the condition previously obtained:

$$c_2 = v_1+m_1.$$[1]

It is interesting to note that only inter-branch flows, i.e. the elements v_1, m_1 and c_2 affect the conditions of equilibrium in the process of reproduction. The remaining elements of the process, c_1, v_2 and m_2 do not affect the equilibrium. An increase in, say, c, increases simultaneously production and demand for the total product of Division 1, i.e. P_1, without disturbing the equilibrium of the process of simple reproduction.

The process of simple reproduction can be presented as a cybernetic diagram shown below:

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[1] More extensive considerations on this subject can be found in the book by O. Lange, *Introduction to Econometrics*, *ed. cit.*, Chapter 3.

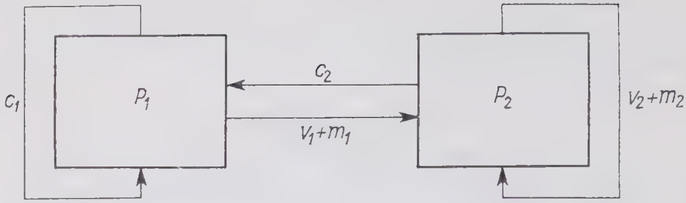

FIG. 2.

It can be seen from this diagram that inter-branch flows in the process of simple reproduction must balance, but the part of the product that remains in a given branch has no effect on the equilibrium of the process.

The conditions of equilibrium are well illustrated by another diagram:

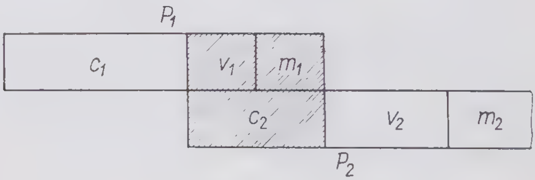

FIG. 3.

The shaded components of the total products, i.e. v_1+m_1 and c_2 equal one another. The remaining components do not affect equilibrium.

Finally, the above mentioned condition of equilibrium of the process of simple reproduction can be presented by using input coefficients. Expressing the components of the total product of both divisions of production, P_1 and P_2, by using input coefficients, we obtain the following table:

$$a_{1c}P_1 + \boxed{a_{1v}P_1 + a_{1m}P_1} = P_1,$$

$$\boxed{a_{2c}P_2} + a_{2v}P_2 + a_{2m}P_2 = P_2.$$

The equilibrium condition $c_2 = v_1 + m_1$ may then be expressed as follows:

$$a_{2c}P_2 = a_{1v}P_1 + a_{1m}P_1,$$

or

$$\frac{P_1}{P_2} = \frac{a_{2c}}{a_{1v}+a_{1m}}. \tag{1a}$$

The condition in this form states that to maintain equilibrium in the process of simple reproduction the ratio of the total product of Division 1 to the total product of Division 2 must equal the ratio of the input coefficient of means of production in Division 2 to the income–product ratio of Division 1, because

$$\frac{Y_1}{P_1} = a_{1v} + a_{1m}.$$

It can also be seen from formula (1a) that for given input coefficients in both divisions, the equilibrium of the process of reproduction depends only upon the ratio of the aggregate products of both divisions, and not upon their absolute size, i.e. the scale of the production plans of the divisions. Thus, for example, a twofold increase of the aggregate products of both divisions does not disturb the existing equilibrium of the process of reproduction.

In this connection the question arises: in which case does the proportion of production of both divisions, i.e. the ratio $\frac{P_1}{P_2}$, change? It follows from formula (1a) that this may happen when the input coefficient of means of production in Division 2, i.e. a_{2c}, changes, or when the income–product coefficient of Division 1, i.e. $a_{1v}+a_{1m}$ changes.

Considering that

$$a_{1c}+a_{1v}+a_{1m} = 1 \quad \text{and} \quad a_{2c}+a_{2v}+a_{2m} = 1,$$

the relationship (1a) can be presented further in a different way

$$\frac{P_1}{P_2} = \frac{a_{2c}}{1-a_{1c}}, \tag{1b}$$

or

$$\frac{P_1}{P_2} = \frac{1-a_{2v}-a_{2m}}{a_{1v}+a_{1m}}. \tag{1c}$$

Finally, starting with the formulae:

$$k_2 = \frac{c_2}{v_2}, \qquad s_1 = \frac{m_1}{v_1},$$

we can write the equilibrium condition of the process of simple reproduction as follows:

$$k_2 v_2 = (1+s_1) v_1,$$

or

$$k_2 a_{2v} P_2 = (1+s_1) a_{1v} P_1.$$

Hence

$$\frac{P_1}{P_2} = \frac{k_2}{1+s_1} \cdot \frac{a_{2v}}{a_{1v}}. \tag{1d}$$

The equilibrium conditions in the form (1a), (or (1b) or (1c), or (1d)) are the starting point for our further considerations and provide a simple tool for analysing the proportions between the division for the production of means of production and the division for the production of means of consumption. This problem has long been a subject of heated discussion in political economy.

We shall now try to determine in a similar way the equilibrium conditions of the process of extended reproduction. As we know, in the process of extended reproduction, the surplus product m is not entirely consumed; the part not consumed constitutes accumulation.

Let us divide the surplus value m in each division into three components: $m = m_c + m_v + m_0$, where m_c denotes the part of accumulation earmarked for increasing constant capital, m_v—the part of accumulation earmarked for increasing variable capital and m_0—the part of the surplus product which is consumed.

The structure of the total product of both production divisions can then be presented as follows:

$$\text{Division 1: } c_1 + v_1 + m_{1c} + m_{1v} + m_{10} = P_1$$
$$\text{Division 2: } c_2 + v_2 + m_{2c} + m_{2v} + m_{20} = P_2.$$

In Marx's schemes, and in later considerations by Lenin, the simplifying assumption has been made to the effect that accumulation is invested in the same branch as that in which it is realized. In reality, however, there are flows of accumulation between branches and, under capitalist free market conditions, they bring about the equalization of the rate of profit. In a planned economy, accumulation takes place primarily in Division 2 and it is invested mainly in Division 1.

If we reject the simplifying assumption that accumulation accrues to the branch in which it has been realized, then the components m_{1c}, m_{1v}, and, similarly, m_{2c} and m_{2v} will be divided into two parts.

Thus, for instance

$$m_{1c} = m_{1c_1} + m_{1c_2},$$

where m_{1c_1} denotes the part of accumulation m_{1c} invested in means of production of Division 1, and m_{1c_2}—the part of accumulation m_{1c} invested in means of production of Division 2.

Analogously

$$m_{1v} = m_{1v_1} + m_{1v_2}.$$

Similarly, also

$$m_{2c} = m_{2c_1} + m_{2c_2},$$

$$m_{2v} = m_{2v_1} + m_{2v_2}.$$

Therefore, the structure of the aggregate product of Division 1 and Division 2 will assume the following form:

$$\text{Division 1} \quad c_1 + m_{1c_1} + m_{1c_2} + \boxed{v_1 + m_{1v_1} + m_{1v_2} + m_{10}} = P_1,$$

$$\updownarrow$$

$$\text{Division 2} \quad \boxed{c_2 + m_{2c_1} + m_{2c_2}} + v_2 + m_{2v_1} + m_{2v_2} + m_{20} = P_2.$$

The equilibrium condition of the process of extended reproduction can be obtained by comparing the requirements of both divisions in means of production with the production of Division 1. Hence we obtain the equation:

$$c_1 + v_1 + m_{1c_1} + m_{1c_2} + m_{1v_1} + m_{1v_2} + m_{10}$$
$$= c_1 + c_2 + m_{1c_1} + m_{1c_2} + m_{2c_1} + m_{2c_2}.$$

After simplifying the foregoing, the following equilibrium condition of extended reproduction is obtained:

$$c_2 + m_{2c_1} + m_{2c_2} = v_1 + m_{1v_1} + m_{1v_2} + m_{10}. \tag{2}$$

This condition means that the equilibrium of the process of extended reproduction is maintained where equilibrium in inter-branch flows has been achieved. When there is equilibrium in flows, it is a matter of indifference where accumulation of a given branch is invested, but it does matter where accumulation has been realized. In this connection, the equilibrium condition (2) can be written in a simplified form:

$$c_2+m_{2c}=v_1+m_{1v}+m_{10}, \tag{2a}$$

Where

$$m_{2c}=m_{2c_1}+m_{2c_2} \quad \text{and} \quad m_{1v}=m_{1v_1}+m_{1v_2}.$$

The equilibrium condition of the process of extended reproduction is presented in the two diagrams below:

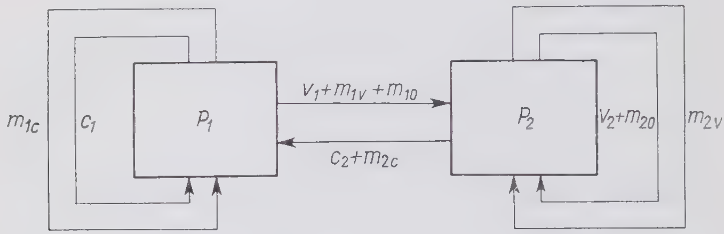

FIG. 4.

FIG. 5.

Whether the equilibrium condition is satisfied, entirely depends upon the equality of the quantities represented by the shaded rectangles. The quantities represented by the unshaded rectangles may be of any size since they do not affect the equilibrium of the process of reproduction. These quantities, however, do affect significantly the rate of economic growth.

The equilibrium condition of extended reproduction can be expressed by the coefficients of inputs and accumulation.

The values of the aggregate products of Division 1 and Division 2, expressed by the input and accumulation coefficients, can be presented as follows:

Division 1: $a_{1c}P_1+\alpha_{1c}P_1+a_{1v}P_1+\alpha_{1v}P_1+\alpha_{10}P_1 = P_1$
Division 2: $a_{2c}P_2+\alpha_{2c}P_2+a_{2v}P_2+\alpha_{2v}P_2+\alpha_{20}P_2 = P_2$.

In the above formulae, we have denoted by a the input coefficients of the means of production (a_c) or of labour (a_v), and by α the coefficients (rates) of accumulation which determine what part of the total product of a given division is accumulated in order to be transformed into means of production (α_c), or to be used for augmenting the wage fund (α_v); the coefficients α_{10} and α_{20} denote, respectively, the part of the total product of Division 1 or Division 2, which is the consumed surplus product.

Thus we have

$$a_{1m} = \alpha_{1c}+\alpha_{1v}+\alpha_{10},$$

$$a_{2m} = \alpha_{2c}+\alpha_{2v}+\alpha_{20}$$

and

$$a_{1c}+\alpha_{1c}+a_{1v}+\alpha_{1v}+\alpha_{10} = 1,$$

$$a_{2c}+\alpha_{2c}+a_{2v}+\alpha_{2v}+\alpha_{20} = 1.$$

The first two of these relationships mean that the share of the surplus product in the total product of a given division equals the sum of the accumulation coefficients and the consumption coefficient of the surplus product.

The equilibrium condition of the process of extended reproduction, (2a) may also be written in the following form:

$$(a_{2c}+\alpha_{2c})P_2 = (a_{1v}+\alpha_{1v}+\alpha_{10})P_1 .$$

Hence

$$\frac{P_1}{P_2} = \frac{a_{2c}+\alpha_{2c}}{a_{1v}+\alpha_{1v}+\alpha_{10}} . \tag{2b}$$

Considering that $a_{1c}+\alpha_{1c}+\alpha_{1v}+\alpha_{1v}+\alpha_{10} = 1$, we may transform formula (2b) as follows:

$$\frac{P_1}{P_2} = \frac{a_{2c}+\alpha_{2c}}{1-a_{1c}-\alpha_{1c}} \tag{2c}$$

The formulae (2b) or (2c) determine the proportions between the value of production in Division 1 and Division 2 (depending upon the input and accumulation coefficients) which ensure equilibrium in the process of reproduction. Thus, formula (2b) states that the process of reproduction can be realized in practice if the ratio of the value of production in Division 1 to the value of production in Division 2 equals the ratio of the sum of the input coefficient of means of production in Division 2 (a_{2c}) and the coefficient of accumulation in this division earmarked for increasing means of production (α_{2c}), to the sum of the labour input coefficient in Division 1 (a_{1v}), the coefficient of accumulation in Division 1 earmarked for augmenting the wage fund (α_{1v}) and the coefficient of consumption of the surplus product in Division 1 (α_{10}).

Let us note further that in the special case when $\alpha_{2c} = \alpha_{1c} = 0$, i.e. in the case of simple reproduction, formula (2c) is transformed into formula (1b) which is the equilibrium condition of the process of simple reproduction.

Analyzing formula (1b), we find that the proportion between the value of production in Division 1 and Division 2, in the case of simple reproduction, depends upon the input coefficients a_{1c} and a_{2c}. The greater they are the greater is the ratio $\frac{P_1}{P_2}$; this means that the share of the total product of Division 1 in the gross national product is then larger.

If the coefficients a_{1c} and a_{2c} decrease, at the same time the ratio $\frac{P_1}{P_2}$ also decreases. If, however, input coefficients a_{1c} and a_{2c} change in the opposite direction, i.e. if a_{1c} grows and a_{2c} declines, (which, in practice, would be a fairly strange phenomenon), then nothing can be said in advance about the direction of change in the ratio $\frac{P_1}{P_2}$.

In the case of extended reproduction, as follows from formula (2c), the proportion between the value of production in Division 1 and Division 2 depends not only on the input coefficients a_{1c} and a_{2c}, but also on the accumulation coefficients α_{1c} and α_{2c}.

The means of production input coefficients a_{1c} and a_{2c} are of a technical character, i.e. they depend exclusively upon the existing state of production techniques. But the coefficients of accumulation earmarked for means of production, i.e. α_{1c} and α_{2c} depend upon economic decisions. Under conditions of a capitalist economy, the quantities α_{1c} and α_{2c} are influenced by numerous decisions of capitalist entrepreneurs, based on the expected rate of return from an investment in new means of production. In a socialist economy the magnitude of the accumulation coefficients α_{1c} and α_{2c} depends upon the economic plan.

If accumulation earmarked for means of production grows in both production divisions, the ratio $\frac{P_1}{P_2}$ increases, and production in Division 1 grows faster than production in Division 2. If accumulation earmarked for means of production in both divisions declines, the ratio $\frac{P_1}{P_2}$ decreases, and thus the share of production of Division 1 in the aggregate production declines.

The situation is more complex if, for instance, α_{2c} grows and simultaneously α_{1c} declines (or vice versa).

It is impossible to tell in advance how this will affect the value of the ratio $\frac{P_1}{P_2}$, but in each specific case this ratio can be calculated on the basis of available statistical data.

The formulae (1b) and (2c) decide, in principle, under what conditions production in Division 1 will grow faster or more slowly than in Division 2. In any case, it follows from these formulae that if accumulation earmarked for means of production in both divisions grows, production P_1 in Division 1 must grow more rapidly than production P_2 in Division 2. Similarly, if the input coefficients earmarked for means of production in both production divisions increase (which means that we are faced with what is known in western literature as the capital intensive type of technical progress), then production P_1 also grows faster than production P_2. If, however, technical progress is of the capital saving type, the ratio $\frac{P_1}{P_2}$ decreases.[2]

[2] All the above conclusions could have been reached without complex mathematical considerations and formulae. It is directly obvious, for instance, that if in Division 2 the demand for means of production increases, production in Division 1 must increase.

As we have mentioned above, the problem of proportions between the value of production in Division 1 and Division 2 is the subject of numerous discussions and controversies. The point is to decide whether the ratio $\frac{P_1}{P_2}$ increases with economic growth.

The traditional view that the ratio $\frac{P_1}{P_2}$ increases all the time, based on the assumption that the share of inputs for means of production a_{1c} and a_{2c} increases constantly, has been questioned, among others, by Professor Bronisław Minc.[3]

Professor Minc states that the outlays on means of production per unit product (i.e. coefficients a_{1c} and a_{2c}, do not necessarily have to grow as a result of technical progress. His contention applies particularly to a socialist economy.

The stand taken by Professor Minc was criticized by A. Pashkov to whom Professor Minc, in turn, replied.[4]

It is clear that this controversy cannot be settled solely on the basis of purely theoretical considerations. Only concrete facts and statistical observations could explain the problem and help to determine whether or not means of production input coefficients actually grow with technical progress.

However, the argument that technical progress which consists in providing the process of production with more and more means of production, i.e. in increasing the amount of technical means per unit of living labour, always results in an increase of the ratio $\frac{P_1}{P_2}$, is not correct. In fact, changes in the ratio $\frac{P_1}{P_2}$ do not depend upon the organic composition of capital, but they depend upon outlays on means of production made in a given production

[3] See Bronisław Minc, *Aktualne zagadnienia ekonomii politycznej socjalizmu* (*Current Problems in the Political Economy of Socialism*, in Polish), Warsaw, 1956, p. 261–304, and *Zagadnienia ekonomii politycznej socjalizmu* (*Problems in the Political Economy of Socialism*, in Polish), Warsaw, 1957, Chapter 5.

[4] See A. Pashkov, Ob odnoi traktovke zakona preimushchestvennogo rosta proizvodstva sredstv proizvodstva, *Voprosy Ekonomiki*, No. 6, 1958, and Bronisław Minc, *Studia i polemiki ekonomiczne* (*Economic Studies and Polemics*, in Polish) Warsaw, 1959, pp. 16–39.

period, i.e. c_1 and c_2, or more exactly, upon the ratio of these parameters to the total product P_1 or P_2.

Indeed

$$a_{1c} = \frac{c_1}{P_1} = \frac{K_1 \mu_1}{P_1}$$

and

$$a_{2c} = \frac{c_2}{P_2} = \frac{K_2 \mu_2}{P_2}.$$

It follows that the magnitude of means of production input coefficients a_{1c} and a_{2c} is determined by constant capital K, replacement coefficient μ and the aggregate product P. The technical composition of the process of production does not directly affect a_{1c} and a_{2c}.

In the process of extended reproduction, the allocation of a part of the total product for accumulation in means of production results in an increase in the ratio $\frac{P_1}{P_2}$, which follows immediately from formula (2c). An increase of the ratio $\frac{P_1}{P_2}$ means, of course, an increased share of the production of means of production P_1 in the gross national product, and vice versa, an increased share of the production of means of consumption in aggregate production entails a reduction of the accumulation coefficients α_{1c} and α_{2c}.

It follows from the above considerations that formula (2c) brings to light the factors which directly affect the production proportions between Division 1 and Division 2.

Formula (2c) can be transformed so that the organic composition of inputs (flows) or the organic composition of capital (stock) appear in it. Although we do not gain much by such a transformation, since the form of formula (2c) is more convenient for an analysis of the factors affecting the ratio $\frac{P_1}{P_2}$, it will be useful if only because in many broad discussions the problem of the proportion between production in Division 1 and in Division 2 is related to the organic composition of capital.

As we know, the organic composition of the inputs (and, therefore, of the flows) in, say, Division 2 is:

$$k_2 = \frac{c_1}{v_2} = \frac{a_{2c} P_2}{a_{2v} P_2} = \frac{a_{2c}}{a_{2v}}.$$

Hence $a_{2c} = k_2 a_{2v}$.

On the basis of the simplifying assumption that accumulation is divided into the additional means of production and the additional wage fund, in proportion to the organic composition of inputs, we obtain:

$$k_2 = \frac{\alpha_{2c}}{\alpha_{2v}}, \quad \text{hence} \quad \alpha_{2c} = k_2 \alpha_{2v}.$$

The coefficients a_{1c} and α_{1c} can be determined analogously. Thus, the formula (2c) will assume the following form:

$$\frac{P_1}{P_2} = \frac{a_{2c}+\alpha_{2c}}{1-a_{1c}-\alpha_{1c}} = \frac{k_2(a_{2v}+\alpha_{2v})}{1-k_1(a_{1v}+\alpha_{1v})}. \tag{3}$$

Formula (3) can be transformed further by introducing in it, instead of the organic composition of inputs, the organic composition of capital (stocks) of Division 1 and Division 2, which we denote by ω_1, and ω_2, respectively.

Since

$$\omega_1 = \frac{K_1}{Z_2} = \frac{c_1 \tau_1}{v_1 \theta_1} = k_1 \frac{\tau_1}{\theta_1},$$

therefore

$$k_1 = \omega_1 \cdot \frac{\theta_1}{\tau_1},$$

and, analogously

$$k_2 = \omega_2 \cdot \frac{\theta_2}{\tau_2}.$$

Thus, formula (3) assumes the form:

$$\frac{P_1}{P_2} = \frac{\omega_2 \frac{\theta_2}{\tau_2}(a_{2v}+\alpha_{2v})}{1-\omega_1 \frac{\theta_1}{\tau_1}(a_{1v}+\alpha_{1v})}. \tag{4}$$

It follows from this formula that if the organic composition of capital ω_1 and ω_2 is constant (*ceteris paribus*) when $\frac{\theta_2}{\tau_2}$ or $\frac{\theta_1}{\tau_2}$ (i.e. the ratio of the period of variable capital turnover to the period of constant capital turnover) increases, the ratio $\frac{P_1}{P_2}$ also increases.

Formula (4) is not very handy, and, therefore, it is more convenient to introduce into it, instead of the periods of turnover of capital θ and τ, their reciprocals, i.e. the periods of replacement. Thus, we obtain the formula:

$$\frac{P_1}{P_2} = \frac{\omega_2 \frac{\mu_{2c}}{\mu_{2v}} (a_{2v} + \alpha_{2v})}{1 - \omega_1 \frac{\mu_{1c}}{\mu_{1v}} (a_{1c} + \alpha_{1v})}. \tag{4a}$$

We infer from formula (4a) that for given organic compositions of capital in both divisions, the ratio $\frac{P_1}{P_2}$ grows with an increasing ratio of the coefficients of replacement of constant and variable capital in Division 1 and Division 2. If, for instance, the period of turnover of the means of production is 30 years, and a transfer is made to less durable means of production with a turnover period of, say, 20 years, thus increasing μ_{1c} and μ_{2c}, the ratio $\frac{P_1}{P_2}$ will also increase. This is explained by the fact that each year more means of production have to be produced for replacement.

We have already pointed out that the most useful formula for analysing the size of the ratio $\frac{P_1}{P_2}$ is formula (2c). The other formulae (3) and (4) or (4a) are more convenient in so far as they contain the organic composition of input or of capital. Generally, it may be said that for studying conditions of inter-branch equilibrium it is most convenient to use input coefficients while the use of structural coefficients leads to more complex formulae. The situation is reversed when we study the process of accumulation; it is then very convenient to use the structural coefficients introduced by Marx.

Let us turn back to the problem of equilibrium in the process of extended reproduction. The formulae expressing the ratio $\frac{P_1}{P_2}$ were introduced on the assumption that the conditions of equilibrium of this process were satisfied. If there is no equilibrium, the proportions may be different than it would appear from the formulae (2c) and (3) or (4); thus, for instance

$$\frac{P_1}{P_2} \neq \frac{a_{2c}+\alpha_{2c}}{1-a_{1c}-\alpha_{1c}}.$$

However, if the equilibrium condition of the process of reproduction is not satisfied, this process cannot be realized. "Bottle-necks" appear in the process of reproduction and production shrinks in proportion to available means which are in relatively short supply. If, for instance, means of production constitute such a bottle-neck and there is a shortage of means of production necessary for a given process of reproduction, this causes a decrease in the production of means of consumption.

Let us denote by λ the ratio $\frac{P_1}{P_2}$ when the condition of equilibrium is satisfied. Thus $\frac{P_1}{P_2} = \lambda$, hence $P_1 = \lambda P_2$. Let us now assume that the equilibrium condition of the process of reproduction does not hold, and let $P_1 < \lambda P_2$. Then, production in Division 2, i.e. P_2, will decline to $\bar{P}_2$, such that $\frac{P_1}{\bar{P}_2} = \lambda$.

If for instance, $\frac{P_1}{P_2} = \frac{3}{4}\lambda$, then P_2 must decrease to $\bar{P}_2$, satisfying the condition $P_1 = \lambda\bar{P}_2$. Therefore, $\frac{3}{4}\lambda P_2 = \lambda\bar{P}_2$, hence $\bar{P}_2 = \frac{3}{4}P_2$ which means that production in Division 2 will decline by 25 per cent.

If, however, means of consumption are a bottle-neck, i.e. the entire available labour force required to use the means of production cannot be employed $\left(\text{i.e. when } P_1 > \lambda P_2, \text{ e.g. } P_1 = \frac{3}{2}\lambda P_2\right)$, then P_1 must again decrease correspondingly which means that in the process of reproduction part of the existing means of production is superfluous.

We can see that, if the equilibrium condition of extended reproduction is not satisfied, the scale of production falls and adjusts itself to the bottle-neck.

As we know, the equilibrium condition of the process of extended reproduction may be presented in the following form (see formula (2a)):

$$m_{2c} = v_1 + m_{1v} + m_{10} - c_2.$$

Adding m_{1c} to both sides of this equation we obtain:

$$m_{1c} + m_{2c} = v_1 + m_{1c} + m_{1v} + m_{10} - c_2.$$

The latter formula can be simplified as follows:

$$m_c = v_1 + m_1 - c_2, \tag{a}$$

where $m_c > 0$.

This condition of equilibrium of the process of extended reproduction was discussed by Marx (without giving any formula) in volume 2 of *Capital*. In the case of simple reproduction, the equilibrium condition is

$$c_2 = v_1 + m_1. \tag{b}$$

From a comparison of the formulae (a) and (b) it appears that extended reproduction occurs when $v_1 + m_1 > c_2$. Then, the difference $v_1 + m_1 - c_2$ is invested in one or both divisions of the economy. The surplus $v_1 + m_1 - c_2$ can be called the accumulation of means of production (it is the net supply of means of production, i.e. the surplus production of means of production over replacement requirements); the left-hand side of equation (a), i.e. m_c, determines the total amount of investment (the demand for additional means of production).

Condition (a) states that, in the process of extended reproduction, accumulation of means of production must equal total investment.

In connection with the above considerations, Nemchinov has introduced the coefficient

$$Q = \frac{m_c}{v_1 + m_1 - c_2}, \tag{5}$$

which he called the *balance coefficient*.

Coefficient Q can also be called *equilibrium coefficient*. When the process of reproduction is in equilibrium, $Q = 1$. If $Q < 1$, then only part of the accumulated means of production is used for investment. In other words, investments are insufficient to absorb the available surplus production. Owing to the existence of surplus accumulation over investments, stocks of non-invested means of production form.[5] The condition $Q > 1$ does not occur in practice. If Q exceeds 1, then in a capitalist economy investment stresses arise; there is inflation and prices of means of production increase.

Nemchinov tried to calculate coefficient Q for Great Britain.[6] On the basis of statistical data mentioned in Chapter 1, Nemchinov discovered that for Great Britain in 1950:

$$v_1+m_1-c_2 = 2663v_1+2010m_1-2196c_2 = 2477,$$

and, at the same time, $m_c = 2143$. Hence

$$Q = \frac{2143}{2477} = 0.865.$$

From these calculations Nemchinov concluded that in Great Britain in 1950 there was a shortage of investment of the order of 13.5 per cent.[7] This means that the national economy could have invested more means of production than it actually did.

It follows from another calculation by Nemchinov[8] that coefficient Q for England in 1935 was 1.24. Thus, this should have been a period of investment stresses.

In a capitalist economy the current production of means of production and thus also the supply of additional means of production depends upon investment decisions made in the past on the basis of expected future profitability of a given branch of

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[5] Of such cases Swedish economists say that "unintentional investments" are created in the form of increased stocks.

[6] V. Nemchinov, *Nekotorye voprosy ispolzovanya balansovego metoda v statistike vzaimnosvyazanykh dynamicheskikh ekonomicheskikh system*, p. 17.

[7] The year 1950 was the period when recession was coming to an end.

[8] See V. Nemchinov, Ob sootnoshenyakh rozshirennogo vosproizvodstva, *Voprosy Ekonomiki*, No. 10, 1958, p. 31.

the economy.[9] In a socialist economy, the magnitude of coefficient Q depends upon the economic plan and, as a rule, this coefficient should equal unity. In Poland, during the 6-year plan period, there was a tendency to fix $Q > 1$, and thus the plans were prepared on too optimistic basis and could never be fully implemented.

In concluding our considerations on the equilibrium conditions of the process of extended reproduction, let us explain one more problem of a terminological nature. If the right-hand side of equation (a), i.e. $v_1+m_1-c_2$ is greater than the left-hand side, i.e. than m_c, then the equilibrium condition is not satisfied because stocks of unnecessary means of production are formed. In western economic literature a different terminology is used although it expresses the same thing. An appraisal of the degree of disequilibrium in the process of reproduction is based on the size of the ratio of investments to *savings*. This is tantamount to studying the ratio $\frac{m_c}{v_1+m_1-c_2}$. If this ratio does not equal unity, then there is no equilibrium. In periods of expansion savings are smaller than investments, and in periods of economic recessions the opposite situation exists: investments are smaller than savings.

This terminology speaking of the lack of equilibrium between investments and "savings" as the source of depressions was originally introduced by Swedish economists and became widely known in western economic literature.

We can also introduce the *condition of disequilibrium* defined as $D = Q-1$. As we can see, D may be either negative or positive. According to the results of Nemchinov's studies, the condition of disequilibrium for Great Britain in 1950 was

$$\bar{D} = Q-1 = 0.865-1 = -0.135 = -13.5\%$$

and in 1935

$$\bar{D} = Q-1 = 1.24-1 = 0.24 = 24\%.$$

In a Marxian analysis of equilibrium of the process of reproduction, the whole production is divided into two specialized divisions. In particular, according to Marx's assumption, means

[9] Compare the mimeographed lectures by O. Lange on: *The Theory of Economic Development*, Chapter 5, The Publishing Department of the University of Warsaw, Warsaw, 1958.

of consumption cannot serve as means of production. In practice, however, different situations arise. In agricultural production, for instance, grain serves simultaneously as a means of consumption and a means of production (grain for sowing and as fodder for livestock). The situation is similar in other branches of production (e.g. coal). For this reason statistical calculations concerning the division of the economy into industries, according to Marx's schemes, are fairly difficult. In our statistics as in those of the Soviet Union, the national economy is divided into divisions A and B, on the basis of the principles outlined in Marxian schemes; but they are not identical with them. In other words, the production Divisions A and B are not identical with the specialized Marxian Divisions 1 and 2. For example, Division A includes the whole armaments production which certainly is not the production of means of production. The whole coal production is included in Division A, while some coal produced is earmarked for consumption, etc.

An additional problem is that of foreign trade. The question arises, for instance, whether export should be included in the means of production or in the means of consumption. An adequate solution of this problem would depend upon what is bought in exchange of exported goods. If, for instance, with the proceeds from the export of bacon we purchase machine tools, then the corresponding part of the bacon production should be included in Division 1. If, on the other hand, with the proceeds from the export of bacon we buy grain, then the exported part of the production of bacon should be included in Division 2.

Thus, if the analysis is to be more precise, we must go beyond the division of the economy into two parts. Even Marx, in Volume 2 of *Capital*, divided Division 2, i.e. the production of consumer goods, into two further subdivisions: (a) the production of means of consumption purchased by the capitalists (which we called luxury goods), and (b) the production of means of consumption purchased by the workers (which he called life necessities).[10]

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[10] What are the equilibrium conditions for an economy divided into three branches? See *Capital*, vol. 2, p. 424 and ff. A similar distinction is often made today in Western economic literature for commodities purchased from wage incomes (wage goods). Strictly speaking, Marx assumes that a part of life

Let us write out a scheme corresponding to the division of the national economy into three branches. It will serve as a transition to a multi-branch scheme of the national economy which we shall discuss in the next chapter:

$$\begin{array}{l} \text{Branch 1: } c_1+v_1+m_{1c}+m_{1v}+m_{10}=P_1 \\ \text{Branch 2: } c_2+v_2+m_{2c}+m_{2v}+m_{20}=P_2 \\ \text{Branch 3: } c_3+v_3+m_{3c}+m_{3v}+m_{30}=P_2 \\ \hline c+v+m_c+m_v+m_0=P \end{array}$$

What are the equilibrium conditions for an economy divided into three branches? Of course, the production of means of production must continue to equal the demand both for the replacement of used up means of production and for increasing their stock. This condition may be written as follows:

$$c_1+v_1+m_{1c}+m_{1v}+m_{10}=c_1+c_2+c_3+m_{1c}+m_{2c}+m_{3c}.$$

Hence, after reduction and transposition we obtain

$$c_2+c_3+m_{2c}+m_{3c}=v_1+m_{1v}+m_{10}. \tag{6}$$

It follows from this formula that, to maintain equilibrium, the surplus production of means of production, not retained in Branch 1, must equal the sum of the flows of production from Branch 1 to Branch 2 and Branch 3. It is easy to check that condition (6) is similar to the equilibrium condition arrived at when the economy is divided into two production divisions. However, when the economy is divided into two divisions, there is only one channel of inter-branch flows; and if the economy is divided into more than two branches, the number of channels is greater. Therefore, if the economy is divided into, say, three branches, in addition to condition (6), we have two further analogous equations:

$$c_2+v_2+m_{2c}+m_{2v}+m_{20}=v_1+v_2+v_3+m_{1v}+m_{2v}+m_{3v}.$$

Hence, after reduction

$$c_2+m_{2c}+m_{20}=v_1+v_3+m_{1v}+m_{3v}, \tag{7}$$

and

$$c_3+v_3+m_{3c}+m_{3v}+m_{30}=m_{10}+m_{20}+m_{30},$$

necessities is purchased by the capitalist. To simplify the exposition, we assume here that they are purchased by the workers only.

or

$$c_3+v_3+m_{3c}+m_{3v}=m_{10}+m_{20}. \tag{8}$$

The first of these two additional conditions, i.e. condition (7), follows from the assumption that production in Branch 2, i.e. the production of means of consumption for the workers equals the sum of the wage fund in all three branches, i.e. $v_1+v_2+v_3$ plus what is earmarked for increasing this wage fund, i.e. $m_{1v}+m_{2v}+m_{3v}$.

Condition (8) is derived from a similar assumption concerning the equality of the production of means of consumption earmarked for the capitalist and the part of the surplus value earmarked for consumption.

It can be shown, however, that conditions (6), (7) and (8) are not independent; in other words, if two of them are satisfied, then the third must also be satisfied. Indeed, deducting both sides of equation (7) from equation (6) we obtain

$$c_3+m_2+m_{3c}-m_{2c}-m_{20}=m_{10}-v_3-m_{3v},$$

and hence

$$c_3+v_3+m_{3c}+m_{3v}=m_{10}+m_{20};$$

which is condition (8).

The interdependence of conditions (6), (7) and (8) follows from the fact that the value of the aggregate production is determined in advance.

Let us consider once more the conditions of equilibrium of the process of reproduction for the case of the division of the economy into three sectors. Condition (6) states that the surplus production in Branch 1 (the right-hand side of equation (6)) equals the demand for the products of Branch 1 forthcoming from the remaining branches. Conditions (7) and (8) may be interpreted in a similar way. On the basis of these comments it is possible to predict what the conditions of equilibrium may be for a larger number of branches. If the national economy is divided into n branches, then there are $n-1$ independent conditions of equilibrium, and each of these conditions expresses the fact that the surplus production in a given branch (i.e. the quantity not consumed by this branch) equals the sum of demands for products of this division forthcoming from the remaining branches.

At first no attention was paid to Marx's two- and three-branch schemes. Only toward the end of the last century has the discussion on this subject begun. Lenin published at that time (1893) the study "On the So-called Market Problem" (W związku z tzw. kwestią rynku)[11] in which he argued against the views expounded by the Narodniks that in Russia the development of capitalism is impossible because there is no market. In this study Lenin used Marx's schemes of reproduction in the analysis of the problem of accumulation and of development of the economy. A little later, the well-known Russian economist, Toukhan-Baranovsky, tried to prove, on the basis of Marxian schemes, that capitalism as an economic system has unlimited possibilities of development. The discussion around these views and on the importance of Marxian schemes of expanded reproduction in asserting the prospects of development of the capitalist system of production lasted 30 years. It has not led to any conclusions because, as it turned out, the schemes of production equilibrium do not suffice for solving the problem which was the subject of this discussion.[12]

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[11] See V. I. Lenin, *Dzieła* (*Works*, in Polish), vol. 1, Warsaw, 1950.

[12] This discussion is described by P. M. Sweezy in: *The Theory of Capitalist Development*, London, 1949 and in the mimeographed lectures by O. Lange, *Teoria rozwoju gospodarczego* (*The Theory of Economic Development*, in Polish), Part I, Chapter 3.

Chapter 3

MULTI-BRANCH SCHEMES OF REPRODUCTION

In the twenties and thirties of this century, interest in Marx's schemes has again been aroused although its source was different from that discussed toward the end of the preceding chapter. New studies have resulted in the development of multi-branch schemes of reproduction.

In the Soviet Union, during the period of drawing up the first 5-year plan (1928–1932), economists began to deal with the problem of the theory of expanded reproduction and accumulation in connection with economic planning and the preparation of socio-economic balance-sheets (balance-sheets of labour, raw materials, personal incomes and expenditures, etc.). Overall compilations of all these various balance-sheets were to provide the information for the general balance-sheet of the whole economy and to serve as a basis for the drawing up of its development plans.

Under socialism, socio-economic balance-sheets began to play a role similar to that played in a capitalist system by economic accounting which makes possible control and provides a basis for new decisions. Similar balance-sheets are in use today also in some capitalist countries. They constitute a further stage in the development of economic accounting and form a basis for social (national) accounting. The development of this form of accounting in capitalist countries was undoubtedly prompted and influenced by the balance-sheet method used in the U.S.S.R.

This was the origin of the system of balanced production and consumption of commodities which came to be known later as the *input–output analysis* or the *inter-branch flows analysis*.

V. Leontief, an American economist of Russian descent, is generally considered to be the founder of the modern input–output analysis. In 1941 he published the study: *The Structure of American Economy 1919–1939*,[1] in which he used and developed the method of input–output analysis for production.[2] Basic ideas for this analysis were conceived in connection with the studies on the balance-sheet of the national economy of the Soviet Union. Leontief, who was then still in the U.S.S.R., published in 1925 a paper entitled: Balance of the National Economy of the U.S.S.R. (Balans narodnogo khozyaistva SSSR) in the journal *Planovoe khozyaistvo*. In this paper he presented the idea of input–output analysis.

A more detailed discussion of input–output analysis can be found in various studies of an economic or econometric nature.[3] We shall, therefore, confine ourselves to a brief description of only the most general features of this method.

Let us assume that the national economy is divided into n branches. Let X_i and $x_i (i = 1, 2, \ldots, n)$ denote the aggregate product and the final product of the i-th branch, respectively, and let $x_{ij} (i, j = 1, 2, \ldots, n)$ stand for reproduction flows from branch i to branch j. Moreover, let X_0 denote the whole labour power, $x_{0i} (i = 1, 2, \ldots n)$ the amount of labour employed in particular branches of production, and x_0—the amount of labour employed outside production or not employed at all.

Let us draw up a balance-sheet of production according to the scheme shown overleaf.[4]

It is evident that there exist many relationships between the quantities shown in this table. First of all, the sum of reproduction flows contained in each row of this table and of the final product

[1] V. V. Leontief, *The Structure of American Economy 1919–1939*, 2nd edition, New York, 1951.

[2] This, incidentally, was not the first study by Leontief in this field. The first paper on this subject was published by him in 1937 in the *Review of Economic Statistics*.

[3] See for instance O. Lange, *Introduction to Econometrics, ed. cit.*, Chapter 3 entitled Theory of Programming. See also P. Sulmicki, *Przepływy międzygałęziowe* (*Inter-branch Flows*, in Polish), Warsaw, 1959 and T. Czechowski, *Wstęp matematyczny do analizy przepływów międzygałęziowych* (*Mathematical Introduction to Inter-branch Flows Analysis*, in Polish), Warsaw, 1958.

[4] Examples of specific balance-sheets of production drawn up according to these schemes can be found in the Appendix to the text-book by O. Lange, *Introduction to Econometrics, ed. cit.*

v_i	x_{01}	x_{02}	...	x_{0n}	x_0	X_0
c_i	x_{11}	x_{12}	...	x_{1n}	x_1	X_1
	x_{21}	x_{22}	...	x_{2n}	x_2	X_2
	.	.		.	.	.
	.	.		.	.	.
	.	.		.	.	.
	x_{n1}	x_{n2}	...	x_{nn}	x_n	X_n
m_i	m_1	m_2	...	m_n		
P_i	X_1	X_2	...	X_n		

equals the total product of a given branch of production. An analogous equation holds for labour. In this way, we obtain $n+1$ *balance-sheet equations for labour and for production in n branches*:

$$X_0 = x_{01}+x_{02}+ \dots +x_{0n}+x_0 = \sum_{j=1}^{n} x_{0j}+x_0$$

$$X_1 = x_{11}+x_{12}+ \dots +x_{1n}+x_1 = \sum_{j=1}^{n} x_{1j}+x_1$$

$$X_2 = x_{21}+x_{22}+ \dots +x_{2n}+x_2 = \sum_{j=1}^{n} x_{2j}+x_2$$

....

$$X_n = x_{n1}+x_{n2}+ \dots +x_{nn}+x_n = \sum_{j=1}^{n} x_{nj}+x_n.$$

These balance sheet equations may be briefly written as follows:

$$X_i = \sum_{j=1}^{n} x_{ij}+x_i \qquad (i=0,1,2,\dots,n). \tag{1}$$

All the quantities appearing in the balance-sheet table can be expressed either in physical units (tons, litres, metres, etc.) or in value units, i.e. in monetary units (e g. in zlotys), or in labour units (e.g. in man-hours). If physical units are used, then only the components in rows can be used and the balance-sheet equations given above can be obtained. If the quantities shown in the table are expressed in value units, then the columns can also be added.

Adding up the components of any column, e.g. of the i-th one, we obtain the cost of production in branch i composed of the labour input x_{0i} and the means of production input $(x_{1i}+x_{2i}+\dots+x_{ni})$. Since the value of the product is, as a rule, greater than the cost of production, a surplus is obtained; it is called the surplus value (profit) m_i. Thus, there exist for each branch *production input equations* (cost equations) which can be expressed as follows:

$$X_i = \sum_{j=1}^{n} x_{ji} + x_{0i} + m_i \qquad (i = 1, 2, \dots, n). \tag{2}$$

The structure of production input equations (2) is the same as that of Marx's schemes. Indeed, we can write $\sum_{j=1}^{n} x_{ji} = c_i$, $x_{0i} = v_i$, and, therefore, the total product of branch i is $X_i = c_i + v_i + m_i$. It can be seen that the production input equations are an extension of the division of the Marxian schemes into n branches.

Leontief makes a very general assumption from a mathematical point of view, namely, that the output of each branch can be used either as a production input in any other branch, or as a means of consumption. It can, of course, also be assumed that some branches of production are of a specific nature, as Marx did in his two- or three-division schemes. For instance, if the output of branch i is not used as a means of production in branch j, then $x_{ij} = 0$. In the case of the division of the national economy into the two specialized Marxian divisions $x_{21} = 0$ and possibly also $x_{22} = 0$ which means that production in Division 2 is not used as a means of production either in Division 1 or in Division 2.

In general, in the matrix (the square table marked by double lines in the balance-sheet) composed of the components x_{ij} $(i, j = 1, 2, \dots, n)$ denoting reproduction flows and the production retained within a given branch (the components x_{ji} located on the diagonal of the flow matrix), some values x_{ij} may equal zero or be so close to zero that, for all intents and purposes, they can be assumed to equal zero.

Let us also note, that all the quantities in Leontief's table are flows and, therefore, their dimension is WT^{-1}. Indeed, X_i, x_i, x_{ij}, etc. denote the quantities or values of output produced or used up during a unit of time, e.g. during 1 year.

Comparing the right-hand sides of equations (2) and (1), we obtain *the equilibrium equations of inter-branch flows*:

$$\sum_{j=1}^{n} x_{ji}+x_{0i}+m_i=\sum_{j=1}^{n} x_{ij}+x_i \quad (i=1, 2, \dots, n). \tag{3}$$

Equations (3) can also be called equations of equilibrium between the demand for, and the supply of, the product of each branch. Indeed, the left-hand side of equations (3) shows what the given branch of production takes and produces and the right-hand side of these equations shows what the branch supplies.

Equations (3) can be transformed by removing from both sides the component x_{ii}; thus we obtain the most frequently used form of the equilibrium equations of inter-branch flows:

$$\sum_{j\neq i}^{n} x_{ji}+x_{0i}+m_i=\sum_{j\neq i}^{n} x_{ij}+x_i \quad (i=1, 2, \dots, n). \tag{3a}$$

There are n equilibrium equations, but only $n-1$ are independent ones because having $n-1$ equilibrium equations we can derive the nth equation.[5]

The meaning of equations (3a) can be expressed as follows. For each branch of production the value of the products that this branch obtains from other branches plus the value of labour and profit equals the value of the products that the branch supplies to other branches of production plus the final product produced in this branch.

It can easily be shown that the general equilibrium equations conform to the condition of equilibrium given by Marx. We shall show this, using the example of the process of simple reproduction.

[5] Cf. the comments on the equilibrium of the process of reproduction when the economy is divided into 3 branches, made at the end of the preceding chapter.

When the economy is divided into two specialized divisions $n = 2$, $x_{21} = 0$ and $x_1 = 0$ (because there is no accumulation in Division 1); the table of the balance-sheet of production is as follows:

x_{01}	x_{02}	x_0	X_0
x_{11}	x_{12}	$x_1 = 0$	X_1
$x_{21} = 0$	x_{22}	x_2	X_2
m_1	m_2		
X_1	X_2		

The equilibrium equation of inter-branch flows (3a) will then assume the following form:

$$x_{01}+x_{11}+x_{21}+m_1 = x_{11}+x_{12}+x_1.$$

If we assume that $x_{21} = 0$ and $x_1 = 0$, we obtain $x_{01}+m_1 = x_{12}$ or, using Marx's notations, $v_1+m_1 = c_1$. This is the well-known Marxian condition of equilibrium for the process of simple reproduction.

We shall now deal in greater detail with the process of expanded reproduction, using the input–output method of analysis. We shall divide the final product of each branch x_i, into two parts: the part earmarked for consumption $x_i^{(0)}$ and the part earmarked for investments I_i. Since, however, the part earmarked for investments may be placed in different branches, we divide, in turn, I_i into the components $I_{i1}, I_{i2}, \ldots, I_{in}$ which denote the amount of the final product of branch i earmarked for investments in branches 1, 2, ..., n, respectively. Thus

$$x_i = x_i^{(0)}+I_i = x_i^{(0)}+I_{i1}+I_{i2}+\ldots+I_{in}.$$

Considering the division of the final product given above, we obtain the following expanded table of the balance-sheet of production inputs and outputs:

$x_{01}\ x_{02}\ \dots\ x_{0n}$	$x'_{01}\ x'_{02}\ \dots\ x'_{0n}$	$x_0^{(0)}$	X_0
$x_{11}\ x_{12}\ \dots\ x_{1n}$ $x_{21}\ x_{22}\ \dots\ x_{2n}$ $\vdots$ $x_{n1}\ x_{n2}\ \dots\ x_{nn}$	$I_{11}\ I_{12}\ \dots\ I_{1n}$ $I_{21}\ I_{22}\ \dots\ I_{2n}$ $\vdots$ $I_{n1}\ I_{n2}\ \dots\ I_{nn}$	$x_1^{(0)}$ $x_2^{(0)}$ $\vdots$ $x_n^{(0)}$	X_1 X_2 $\vdots$ X_n
matrix of reproduction flows	matrix of investment flows		

In statistical practice, it is possible to divide inter-branch flows into flows earmarked for reproduction and investment, but the task is undoubtedly fairly cumbersome. Let us note that the quantities located on the diagonal of the matrix of investment flows, i.e. $I_{11}, I_{22}, \dots, I_{nn}$ constitute the investments of the final product in its own branch, i.e. in the branch in which the final product had been produced. In the above table, we have also divided labour into x_{0i}, i.e. labour employed in reproduction, and x'_{0i}, i.e. labour employed in investments.

Equation (3a) will assume the following form:

$$\sum_{j\neq 1}^{n} x_{ji}+x_{0i}+m_i = \sum_{j\neq 1}^{n} x_{ij}+\sum_{j=1}^{n} I_{ij}+x_i^{(0)} \qquad (i=1,2,\dots,n). \tag{3b}$$

In the case of the two-division Marxian scheme, the table of the expanded input–output balance sheet is as follows:

x_{01}	x_{02}	x'_{01}	x'_{02}	$x_0^{(0)}$	X_0
x_{11} 0	x_{12} 0	I_{11} 0	I_{12} I_{22}	0 $x_2^{(0)}$	X_1 X_2
m_1	m_2				
X_1	X_2				

The equilibrium condition in this case is:

$$x_{01}+m_1 = x_{12}+I_{11}+I_{12}\,,$$

and using the notations introduced by Marx we have:

$$v_1+m_1 = c_2+m_{1c}+m_{2c}\,.$$

Considering further that $m_1 = m_{1c}+m_{1v}+m_{10}$, the last equation can easily be transformed into the known equilibrium condition of the process of expanded reproduction:

$$v_1+m_{1v}+m_{10} = c_2+m_{2c}.$$

We shall now calculate the aggregate product of the national economy (the gross national product)

$$X = \sum_{i=1}^{n} X_i\,.$$

We shall use equations (3) whose left- and right-hand sides represent X_i, i.e. the total product of branch i. We obtain:

$$X = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} + \sum_{i=1}^{n} x_{0i} + \sum_{i=1}^{n} m_i = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} + \sum_{i=1}^{n} x_i\,.$$

Let us note, first of all, that the double sums appearing on both sides of the last equation equal each other, and can therefore be reduced; $\sum_{i=1}^{n} x_{0i}$ constitutes the total amount of labour employed in production and equals v, $\sum_{i=1}^{n} m_i$ is the total surplus value m and $\sum_{i=1}^{n} x_i$ constitutes the net product of the national economy which we denote by x.

Hence, equation: $v+m = x$ which expresses the known theorem that the national income $v+m$ (the sum of wages and surplus value) equals the net product x, i.e. that part of the aggregate product which has not been used up for reproduction.[6] In particular branches of the national economy this condition may not hold, but it is satisfied in so far as the whole national economy is concerned.

[6] Let us remember that in statistics the sum $v+m$ is called value added. Thus, the value added for the whole economy equals the net product.

Let us now analyse production balance-sheet equations (1). We shall introduce into our considerations *the technical coefficients of production* a_{ij} defined as follows:

$$a_{ij} = \frac{x_{ij}}{x_j} \quad (i, j = 1, 2, \ldots, n).$$

The technical coefficient of production a_{ij} tells us what amount (or what value) of the product of branch i is necessary to produce a unit (in physical or value terms) of the product of branch j.[7] If the coefficient a_{ij} is determined as the ratio of x_{ij} to X_j expressed in value-monetary units, then a_{ij} is called *the input coefficient.*

It follows from the definition of the technical coefficients of production that inter-branch flows can be determined from the relevant technical coefficients (input coefficients) and from the size of the total products of particular branches.

Indeed, $x_{ij} = a_{ij}X_j(i, j = 1, 2, \ldots, n)$, and then the balance-sheet equations (1) can be presented in a developed form as follows:

$$\begin{cases} X_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + x_1, \\ X_2 = a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n + x_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_n = a_{n1}X_1 + a_{n2}X_2 + \ldots + a_{nn}X_n + x_n, \end{cases} \tag{4}$$

These equations can also be presented in the following form:

$$\begin{cases} (1-a_{11})X_1 - a_{12}X_2 - \ldots - a_{1n}X_n = x_1, \\ -a_{21}X_1 + (1-a_{22})X_2 - \ldots - a_{22}X_n = x_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ -a_{n1}X_1 - a_{n2}X_2 - \ldots + (1-a_{nn})X_n = x_n. \end{cases} \tag{5}$$

The system of equations (5) consists of n linear equations containing $2n$ unknowns: $X_1, X_2, \ldots, X_n$, and $x_1, x_2, \ldots, x_n$. If we assume that the final products $x_1, x_2, \ldots, x_n$ are determined (e.g. by the economic plan), then the remaining unknowns $X_1, X_2, \ldots, \ldots, X_n$, i.e. the total products, can be determined from the system of equations (5).

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[7] A more detailed discussion of the technical coefficients of production and the methods of solving a system of balance-sheet equations (with respect to x_i or X_i) can be found in Chapter 3 of *Introduction to Econometrics.*

The condition for obtaining a universal solution of the linear equations (5) is that the determinant of this system of equations shall not equal zero, i.e.

$$D = \begin{vmatrix} (1-a_{11}) & -a_{12} & \dots & -a_{1n} \\ -a_{21} & (1-a_{22}) & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & (1-a_{nn}) \end{vmatrix} \neq 0.$$

By solving the system of equations (5) we shall be able to analyse more closely the relations (proportions) that must exist between the total products and the final products of various branches.

Moreover, if we divide the final products appearing on the right-hand side of the system of equations (5) into parts earmarked for consumption and investments:

$$\left| \begin{array}{l} x_1 = k_1 x_1 + l_1 x_1, \\ x_2 = k_2 x_2 + l_2 x_2, \\ \dots\dots\dots\dots \\ x_n = k_n x_n + l_n x_n, \end{array} \right|$$

where the coefficients k_i and l_i denote, respectively, the share of consumption and investments in the final products, it will become apparent that the proportions obtaining for the total products and the final products also depend upon these coefficients of consumption and investments.

In this way, we obtain a method which enables us to study the proportions between the total products of particular branches when the national economy is divided into any number of branches and not only into two divisions, as in the case discussed in the preceding chapter.

Assuming that the matrix of production technique

$$\begin{Vmatrix} 1-a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & 1-a_{22} & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & 1-a_{nn} \end{Vmatrix}$$

is of rank n which means that the determinant D composed of the components of this matrix does not equal zero,[8] the solution of the system of linear equations (5) with respect to the total products $X_1, X_2, \dots, X_n$ can be presented as follows:[9]

$$X_i = \frac{\sum_{j=1}^{n} D_{ji} x_j}{D} \qquad (i = 1, 2, \dots, n),$$

or

$$X_i = \sum_{j=1}^{n} A_{ij} x_j \qquad (i = 1, 2, \dots, n). \tag{6}$$

where the quantities $A_{ij} = \frac{D_{ji}}{D}$ are the components of a matrix inverse[10] to the matrix of production technique.

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[8] If it is found that $D = 0$, then, depending upon the rank of the matrix of production technique, the size of one or several total products could be chosen arbitrarily. If, for instance, the matrix is of rank $n-1$ (i.e. not all minors with $n-1$ rows and $n-1$ columns of this matrix equal zero), one of the quantities $X_1, X_2, \dots, X_n$ can be chosen arbitrarily and the remaining ones will be determined by the system of equations (5).

[9] We use here the Cramer rule for solving a system of linear equations.

[10] The inverse matrix A^{-1} with respect to a given square matrix $A = \|a_{ij}\|$ is obtained by forming from co-factors D_{ij}, components a_{ij}, determinant D of matrix A, the matrix:

$$\left\| \begin{matrix} D_{11} & D_{12} & \dots & D_{1n} \\ D_{21} & D_{22} & \dots & D_{2n} \\ \dots & \dots & \dots & \dots \\ D_{n1} & D_{n2} & \dots & D_{nn} \end{matrix} \right\|$$

and then by dividing the components of the matrix thus obtained by determinant D and transposing rows in place of columns and vice versa:

$$A^{-1} = \left\| \begin{matrix} \frac{D_{11}}{D} & \frac{D_{21}}{D} & \dots & \frac{D_{n1}}{D} \\ \frac{D_{12}}{D} & \frac{D_{22}}{D} & \dots & \frac{D_{n2}}{D} \\ \dots & \dots & \dots & \dots \\ \frac{D_{1n}}{D} & \frac{D_{2n}}{D} & \dots & \frac{D_{nn}}{D} \end{matrix} \right\|.$$

Formula (6) can be written in a developed form:

$$X_i = A_{i1}x_1 + A_{i2}x_2 + \ldots + A_{ik}x_k + \ldots + A_{in}x_n \qquad (i = 1, 2, \ldots, n).$$

It follows that the total product of branch i is the weighted sum of all final products. The weights A_{ik} appearing in this sum indicate by how much total production in branch i (e.g. coal) should be increased if the final product of branch k (e.g. steel) is to increase by one unit. Indeed

$$\frac{\partial X_i}{\partial x_k} = A_{ik},$$

which means that, if the increase in x_k equals 1, the increase in X_i equals A_{ik}.

The coefficients A_{ik} are called *the coefficients of additional requirements*, or *the product intensive coefficients*, in a broad sense of this word, since it may also mean an increase in, say, the amount and value of machinery for raising the output of coal.

The product intensive coefficients (and the labour-output coefficients formed in a similar way) are analogous to multipliers used by Keynes in his studies of the effect of an increase in the production of consumer goods on employment.[11]

Having the solution of the system of equations (5) for unknowns $X_1, X_2, \ldots, X_n$, we can determine the proportion existing between the total products of any two branches of production "i" and "k":

$$\frac{X_i}{X_k} = \frac{\sum_{j=1}^{n} A_{ij}x_j}{\sum_{j=1}^{n} A_{kj}x_j} = \frac{A_{i1}x_1 + A_{i2}x_2 + \ldots + A_{in}x_n}{A_{k1}x_1 + A_{k2}x_2 + \ldots + A_{kn}x_n}. \qquad (7)$$

We shall show now that the general formula (7) which determines the proportions between the total products of any two branches conforms—for the special case of Marx's scheme in which two specialized divisions are considered—to the formula:

$$\frac{P_1}{P_2} = \frac{a_{2c} + \alpha_{2c}}{1 - a_{1c} - \alpha_{1c}},$$

[11] A more extensive treatment of product intensive coefficients can be found in the book by O. Lange, *Introduction to Econometrics, ed. cit.*, Chapter 3.

which was derived in the preceding chapter (Chapter 2, formula (2c)).

The latter formula, determining the proportion between the value of production in Division 1 and Division 2, can be written, after applying the more general notations used in this chapter, in the following way:

$$\frac{X_1}{X_2} = \frac{a_{12}+\alpha_{12}}{1-a_{11}-\alpha_{11}}. \tag{8}$$

Indeed, the symbol a_{2c} tells us what part of the total production in Division 2 constitute the means of production produced in Division 1 and used up in Division 2. Thus, a_{2c} is the coefficient of the input of the product obtained in Division 1 and used up in Division 2 which, according to notations now used, is denoted by symbol a_{12}.

Similarly, coefficient α_{2c} denotes the part of the total product of Division 1 earmarked for investments in Division 2; this we now denote by symbol α_{12}, etc.

Formula (2) confirms the obvious fact that the greater the amount of the means of production needed either by Division 1 (i.e. the greater a_{11} or α_{11}) or by Division 2 (i.e. the greater a_{12} or α_{12}), the greater the ratio $\frac{X_1}{X_2}$, i.e. the greater the share of the output of means of production in the total product.

In the case of two divisions ($n = 2$), the table of reproduction flows is correspondingly simplified and the determinant of the matrix of production technique is:

$$D = \begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix}.$$

The solutions of the system of balance equations (only two in this case) for X_1 and X_2 are as follows:

$$X_1 = \frac{(1-a_{22})\,x_1+a_{12}x_2}{D}, \qquad X_2 = \frac{a_{21}\,x_1+(1-a_{11})\,x_2}{D}.$$

The ratio of the aggregate products of Division 1 and Division 2 is determined by the following formula:

$$\frac{X_1}{X_2} = \frac{(1-a_{22})\,x_1+a_{12}\,x_2}{a_{21}\,x_1+(1-a_{11})\,x_2}. \tag{9}$$

In Marx's schemes it is assumed that these divisions are specialized, namely, that the product of Division 2 does not constitute a means of production in either of the two Divisions, and, therefore, $x_{21} = 0$ and $x_{22} = 0$; and in consequence also $a_{21} = 0$ and $a_{22} = 0$. On this assumption, formula (9) is simplified and assumes the following form:

$$\frac{X_1}{X_2} = \frac{x_1 + a_{12} x_2}{(1 - a_{11}) x_2}. \tag{10}$$

Let us now consider separately the case of simple reproduction and the case of expanded reproduction.

In simple reproduction there is no accumulation and therefore the coefficients of accumulation $\alpha_{11} = 0$ and $\alpha_{12} = 0$; moreover the whole product of Division 1 is used up for replacement, and thus that final product of Division 1 $x_1 = 0$. It follows that formulae (8) and (10) assume an identical form:

$$\frac{X_1}{X_2} = \frac{a_{12}}{1 - a_{11}}.$$

In the case of expanded reproduction $x_1 > 0$ and $x_1 = \alpha_{11} X_1 + \alpha_{12} X_2$, where α_{11} denotes what fraction of the total production of Division 1 is invested in Division 1; similarly α_{12} denotes the fraction of the total production of Division 2 composed of the products of Division 1 invested in Division 2.

Moreover, in view of the specialization of the divisions there are no flows from Division 2 to Division 1 and the whole production in Division 2 is consumed; thus $x_2 = X_2$.

Therefore, formula (10) can be transformed in the following way:

$$\frac{X_1 (1 - a_{11}) x_2}{X_2} = x_1 + a_2 x_2,$$

therefore

$$\frac{X_1 (1 - a_{11}) x_2}{X_2} = \alpha_{11} X_1 + \alpha_{12} X_2 + a_{12} x_2,$$

and hence (since $x_2 = X_2$)

$$X_1 (1 - a_{11}) = \alpha_{11} X_1 + \alpha_{12} X_2 + a_{12} X_2,$$

or

$$X_1(1-a_{11}-\alpha_{11}) = (a_{12}+\alpha_{12})X_2,$$

and therefore

$$\frac{X_1}{X_2} = \frac{a_{12}+\alpha_{12}}{1-a_{11}-\alpha_{11}}$$

which is formula (2).

We have shown that multi-branch reproduction schemes and formula (7) which gives the ratio between the total products of any two branches of reproduction are an extension of the Marxian schemes to cover any number of branches. If, then, the number of branches is limited to two and if it is assumed, after Marx, that these branches are specialized, we obtain the formula for the ratio of the total products of Division 1 and Division 2 which we have derived earlier directly from the Marxian schemes.

CHAPTER 4

INFLUENCE OF INVESTMENT ON THE GROWTH OF PRODUCTION

IN the preceding chapters we dealt with the problem of reproduction and accumulation from the static point of view and assumed equilibrium conditions in the process of reproduction. We shall now appraise the influence of accumulation and investment on the growth of production.[1]

Let us recall that by accumulation we understand that part of the final product which is not consumed. Investment consists in using up this part of the final product as a means of production, thus returning it into the process of production.

By accumulation and investment we increase the amount of means of production and the total product in the subsequent production periods.

Thus, the basis for accumulation and investment is provided by the portion of the final product which is not consumed but saved. Denoting by x_i the final product of branch i, by $x_i^{(0)}$ the part consumed and by I_i the invested part of the final product of branch i, we obtain the following equation for each branch of production:[2]

$$x_i = x_i^{(0)} + I_i \quad (i = 1, 2, \ldots, n).$$

The part not consumed I_i of the final product of branch i can be invested in any branch of production and therefore:

$$I_i = I_{i1} + I_{i2} + \ldots + I_{in} = \sum_{j=1}^{n} I_{ij} \quad (i = 1, 2, \ldots, n),$$

where I_{ij} denotes the product of branch i invested in branch j.

[1] This problem was presented by the author in the paper: Model wzrostu gospodarczego (A Model of Economic Growth) *Ekonomista*, No. 3, 1959.

[2] We exclude from our considerations the formation or depletion of existing stocks and the influence of imports and exports on the size of investments and on the growth of production.

Let us construct *an expanded input–output balance-sheet table* including both reproduction and investment flows.

x_{11}	x_{12}	...	x_{1n}	I_{11}	I_{12}	...	I_{1n}	$x_1^{(0)}$	X_1
x_{21}	x_{22}	...	x_{2n}	I_{21}	I_{22}	...	I_{2n}	$x_2^{(0)}$	X_2
.	.		.	.	.		.	.	.
.	.		.	.	.		.	.	.
.	.		.	.	.		.	.	.
x_{n1}	x_{n2}		x_{nn}	I_{n1}	I_{n2}	...	I_{nn}	$x_n^{(0)}$	X_n
Table (matrix) of reproduction flows				Table (matrix) of investment flows					

Let us note that on the basis of the matrices of reproduction and investment flows we can construct a matrix of aggregate flows; the latter is equal to the sum of the two former matrices:

$$\left\| \begin{matrix} x_{11}+I_{11} & x_{12}+I_{12} & \dots & x_{1n}+I_{1n} \\ x_{21}+I_{21} & x_{22}+I_{22} & \dots & x_{2n}+I_{2n} \\ \dots & \dots & \dots & \dots \\ x_{n1}+I_{n1} & x_{n2}+I_{n2} & \dots & x_{nn}+I_{nn} \end{matrix} \right\| =$$

$$= \left\| \begin{matrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{matrix} \right\| + \left\| \begin{matrix} I_{11} & I_{12} & \dots & I_{1n} \\ I_{21} & I_{22} & \dots & I_{2n} \\ \dots & \dots & \dots & \dots \\ I_{n1} & I_{n2} & \dots & I_{nn} \end{matrix} \right\| .$$

The balance equations, obtained on the basis of the expanded inter-branch flows table, have the following form:

$$X_i = \sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} I_{ij} + x_i^{(0)} \quad (i = 1, 2, \dots, n). \tag{1}$$

This can also be written as follows:

$$X_i = \sum_{j=1}^{n} (x_{ij} + I_{ij}) + x_i^{(0)}. \tag{1a}$$

The amount of the product of branch i that should be invested in branch j in order to obtain in the latter a specified increase in total production is determined by the technical conditions of production. In this connection we shall introduce the notion of *investment coefficients* b_{ij} as defined by formula:

$$b_{ij} = \frac{I_{ij}}{\Delta X_j}, \tag{2}$$

in which ΔX_j denotes the increase in the total production of branch j. The investment coefficients are purely technical indicators since their magnitude depends exclusively upon the techniques of production. They indicate how much of the product of branch i should be invested in branch j to increase the total production of this branch by one unit.

If the growth of the total product of a given branch is proportional to the investment flow from another branch, (as is usually assumed), then the investment coefficient b_{ij} equals the *capital–output ratio* c_{ij}.

To show that this is so, we denote by K_{ij} the amount of the product of branch i engaged (in the form of a stock of means of production, i.e. in the form of constant capital) in branch j; this is the amount of constant capital in branch j in the material form of products of branch i. The capital–output ratio c_{ij} is then defined by the formula

$$c_{ij} = \frac{K_{ij}}{X_j}, \tag{3}$$

and hence

$$K_{ij} = c_{ij} X_j .$$

The capital K_{ij} and the total product X_j change during the process of production and thus are functions of time t; therefore:

$$K_{ij}(t) = c_{ij} X_j(t).$$

If we assume that the process of growth of capital $K_{ij}(t)$ and of the total product $X_j(t)$ is continuous and therefore these quantities are continuous differentiable functions, then the last equality can be differentiated for variable t. Hence, we obtain

$$\frac{dK_{ij}}{dt} = c_{ij} \frac{dX_j}{dt}. \tag{4}$$

The derivative appearing on the left-hand side of formula (4) denotes the increase in constant capital at any given moment or (approximately) the increase in constant capital per unit time (e.g. per year), and therefore $\frac{dK_{ij}}{dt} = I_{ij}$. Similarly, the derivative $\frac{dX_j}{dt}$ represents approximately the increase in the product per unit time.[3]

Equation (4) can then be written in the form:

$$I_{ij} = c_{ij} \Delta X_j. \tag{4a}$$

Comparing equation (2) with equation (4a), we see that $b_{ij} = c_{ij}$, i.e. investment coefficients b_{ij} equal the corresponding capital–output ratios c_{ij}.

In studying the process of reproduction and investment we use two kinds of coefficients: the technical coefficients of production (more exactly of "reproduction") $a_{ij} = \frac{x_{ij}}{X_j}$, and the investment coefficients $b_{ij} = \frac{I_{ij}}{X_j}$. The first of these two coefficients determines the amount of the product of branch i used up for producing a unit of the total product in branch j; the second coefficient determines the amount of the product of branch i required for producing an additional unit of the total product in branch j. This may be, for instance, the amount of steel needed to produce an *additional* unit of a textile product. The former are the coefficients expressing the technical conditions of *using up* means of production in current production, the latter are the coefficients expressing the technical conditions of *the growth* of production.[4]

At a first glance it might appear that the coefficients a_{ij} and b_{ij} actually denote the same thing and equal one another. How-

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[3] If we assume that K_{ij} and X_j are linear functions with respect to t, then, exactly, $\frac{dK_{ij}}{dt} = I_{ij}$ and $\frac{dX_j}{dt} = \Delta X_j$ (per unit of time).

[4] A more detailed discussion on this subject can be found in the paper by O. Lange, Produkcyjno-techniczne podstawy efektywności inwestycji (Production and Technical Basis of the Effectiveness of Investment), *Ekonomista*, No. 6, 1958, pp. 1166–67.

ever, generally, this is not so. The equality $a_{ij} = b_{ij}$ occurs only if the period of turnover, i.e. the lifespan of an investment, is 1 year. If, however, the lifespan of an investment is longer than 1 year and *the period of its turnover* is T_{ij}, then the investment coefficient:

$$b_{ij} = a_{ij} T_{ij}. \tag{5}$$

Indeed, if we want to increase for instance the production of textiles in the following year, we must invest in a whole machine, although in one year only part of the machine equal to $\frac{1}{T_{ij}}$ of its value is used up. Here $T_{ij} > 1$. If, on the other hand, the period of turnover of an investment $T_{ij} < 1$, that is, if the investment in question is for instance an additional input of a raw material which is used up within one month, i.e. $T_{ij} = \frac{1}{12}$, to produce a unit of product it suffices to invest $\frac{1}{12}$ of the amount used up during one year. Let us assume, for example, that to produce 1 metre of some fabric 1 kilogram of cotton is used up during one quarter $\left(T_{ij} = \frac{1}{4}\right)$; in this case, to increase the production of this fabric by 1 metre during 1 year an additional quantity of cotton equal to 1 kg $\frac{1}{4} = \frac{1}{4}$ kg will suffice.

It follows from the condition that the investment coefficient b_{ij} must be equal to the capital–output ratio, that $b_{ij} = \frac{K_{ij}}{X_j}$, and therefore[5] $K_{ij} = b_{ij} X_j$.

Differentiating the last equation in which the variables K_{ij} and X_j are regarded as continuous (and differentiable) functions of time t, we obtain:

$$\frac{dK_{ij}}{dt} = b_{ij} \frac{dX_j}{dt},$$

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[5] It is tacitly assumed that the increase in production is proportional to the amount of constant capital additionaly invested.

and since the derivative $\frac{dK_{ij}}{dt}$ is the increment of constant capital per unit of time, i.e. the investment I_{ij}, then:[6]

$$I_{ij} = b_{ij} \frac{dX_j}{dt}. \qquad (6)$$

Formula (6) means that the size of investment I_{ij} (i.e. the investment of the product of branch i in branch j) is proportional to the velocity at which the total product of branch j increases at a given moment (the author distinguishes between the velocity and the actual rate of growth, see further).

Formula (6) is more convenient than formula: $I_{ij} = b_{ij}\,\Delta X_j$ in which the size of investment I_{ij} depends upon the increase ΔX_j and, therefore, also upon the period for which this increase has been determined.[7]

If instead of investment flows I_{ij} we introduce their values, defined by formula (6), we can write the expanded table of reproduction and investment flows in the following way:

$a_{11}X_1 \quad a_{12}X_2 \dots a_{1n}X_n$	$b_{11}\frac{dX_1}{dt}\, b_{12}\frac{dX_2}{dt} \dots b_{1n}\frac{dX_n}{dt}$	$x_1^{(0)}$	X_1
$a_{21}X_1 \quad a_{22}X_2 \dots a_{2n}X_n$	$b_{21}\frac{dX_1}{dt}\, b_{22}\frac{dX_2}{dt} \dots b_{2n}\frac{dX_n}{dt}$	$x_2^{(0)}$	X_2
....	....	...	...
$a_{n1}X_1 \quad a_{n2}X_2 \dots a_{nn}X_n$	$b_{n1}\frac{dX_1}{dt}\, b_{n2}\frac{dX_2}{dt} \dots b_{nn}\frac{dX_n}{dt}$	$x_n^{(0)}$	X_n

[6] The same result can be obtained in a different way. We know that $I_{ij} = b_{ij}\Delta X_j$, and ΔX_j represents the increase in production per unit time: $\Delta t = 1$. If $\Delta t \neq 1$, then $I_{ij} = b_{ij}\frac{X_j}{\Delta t}$. Assuming that $\Delta t \to 0$ and using the notion of limit, we obtain $I_{ij} = b_{ij}\frac{dX_j}{dt}$.

[7] There is an analogy, here, with the definition of velocity of motion v, either as the ratio of increase in the path s to the increase in time t: $v = \frac{\Delta s}{\Delta t}$, or as the derivative of the path with respect to time:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}.$$

It is obvious that the second definition is more convenient and more general.

In consequence, the balance equations (1) will assume the following form:

$$X_i(t) = \sum_{j=1}^{n} a_{ij} X_j(t) + \sum_{j=1}^{n} b_{ij} \frac{dX_j(t)}{dt} + x_i^{(0)}(t)$$

$$(i = 1, 2, \ldots, n). \qquad (7)$$

This is a system of n differential equations of the first order in which functions: $X_1(t), X_2(t), \ldots, X_n(t)$, and functions $x_1^{(0)}(t)$, $x_2^{(0)}(t), \ldots, x_n^{(0)}(t)$ appear as unknowns. They are continuous functions of time t by assumption.

The unknowns $x_1^{(0)}(t), x_2^{(0)}(t), \ldots, x_n^{(0)}(t)$, which determine the parts of the final product allocated for consumption at a given moment, may be eliminated from the system of equations (7) if we assume that the rates of gross investment α_i or the rates of consumption k_i are given for each branch of production.

The rate of investment is determined by the formula:[8]

$$\alpha_i = \frac{X_i(t) - x_i^{(0)}}{X_i(t)}. \qquad (8)$$

This is the ratio of the surplus total product of a given branch, over its consumption, to the total product, i.e. the fraction determining what part of the total product of a given branch is used for reproduction and investment in the same or in other branches.

The rate of consumption $k_i = \frac{x_i^{(0)}}{X_i(t)}$ indicates what part of the total product of a given branch is consumed.

It follows from the definition of the rate of gross investment and of the rate of consumption that $\alpha_i = 1 - k_i$.

It follows from equation (8) that $X_i(t) - x_i^{(0)}(t) = \alpha_i X_i(t)$, and therefore the system of equations (7), containing $2n$ unknown

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[8] In our further considerations we assume that the rates of gross investment are constant. In reality they are subject to change and, therefore, are functions of time t and should be written in the form $\alpha_i(t)$.

functions, can be transformed into a system of equations with n unknown functions:

$$\alpha_i X_i(t) = \sum_{j=1}^{n} a_{ij} X(t) + \sum_{j=1}^{n} b_{ij} \frac{dX_j(t)}{dt} \qquad (i = 1, 2, \ldots, n), \qquad (7a)$$

or

$$-\alpha_i X_i(t) + \sum_{j=1}^{n} a_{ij} X_j(t) + \sum_{j=1}^{n} b_{ij} \frac{dX_j(t)}{dt} = 0 \qquad (i = 1, 2, \ldots, n). \qquad (7b)$$

Equations (7b) constitute a system of n homogeneous linear differential equations[9] of the first order with constant coefficients containing n unknown functions and its derivatives. This kind of system of differential equations can be solved by a relatively simple method of "trial and error substitutions" often used for solving differential equations.

Since single differential equations of the first order and with constant coefficients have solutions in the form of an exponential function, we try to ascertain if solutions of the system of differential equations of the type (7b) are exponential functions of the type $X_i(t) = k_i e^{vt}$. To check this assumption, we substitute in the system of equations (7b) the function $k_i e^{vt}$ for $X_i(t)$. We obtain the following system of equations:

$$-\alpha_i k_i e^{vt} + \sum_{j=1}^{n} a_{ij} k_j e^{vt} + \sum_{j=1}^{n} b_{ij} k_j v e^{vt} = 0 \quad (i = 1, 2, \ldots, n) \quad (7c)$$

or, after dividing both sides of the equation[10] by e^{vt}:

$$-\alpha_i k_i + \sum_{j=1}^{n} a_{ij} k_j + \sum_{j=1}^{n} b_{ij} k_j v = 0 \qquad (i = 1, 2, \ldots, n). \qquad (7d)$$

It is found that the system of differential equations (7b) is satisfied by the solutions: $X_i = k_i e^{vt}$ $(i = 1, 2, \ldots, n)$ if the values of k_i and the parameter v are so chosen that the system of

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[9] A differential equation is called homogeneous if it does not have free terms, i.e. terms not containing an unknown function or its derivatives.

[10] The exponential function e^{vt} cannot equal zero for any value of t.

homogeneous linear algebraic equations (7d) is satisfied. We know that a system of homogeneous linear algebraic equations is satisfied (except for a trivial case when all roots equal zero) only when the determinant of this system equals zero, and then:

$$\begin{vmatrix} -\alpha_1+a_{11}+b_{11}v & a_{12}+b_{12}v & \dots & a_{1n}+b_{1n}v \\ a_{21}+b_{21}v & -\alpha_2+a_{22}+b_{22}v & \dots & a_{2n}+b_{2n}v \\ \dots & \dots & \dots & \dots \\ a_{n1}+b_{n1}v & a_{n1}+b_{n2}v & \dots & -\alpha_n+a_{nn}+b_{nn}v \end{vmatrix} = 0 \qquad (9)$$

Let us note that in this determinant the components located on the main diagonal are of a different type from the remaining components of the determinant. This is so because it is only in the components located on the main diagonal that there appear the rates of gross investment α_i in addition to the technical coefficients a_{ii} and the investment coefficients b_{ii}.

Condition (9) is the characteristic equation of the system of differential equations (7b). From the characteristic equation (9), we can determine v and then calculate the values of k_i from equation (7d).

Let us note that the determinant (9), after it is expanded, forms a polynomial of order n for variable v. Thus, condition (9) may be presented in the form of the following equation:

$$A_n v^n + A_{n-1} v^{n-1} + \dots + A_{1v} + A_0 = 0. \qquad (10)$$

Equation (10) of order n with respect to v has, as we know, n roots (some of them may be multiple i.e. may have the same value), real or complex; as complex roots always appear in pairs, to each complex root there corresponds another complex root, conjugate with it.[11]

Substituting successively in the system of equations (7d) the roots $v_1, v_2, \dots, v_n$ from equation (10), we obtain n systems of equations with n unknowns. Each of these n systems gives us a solution for the unknowns k_i. Thus, for each k_i we have n values corresponding to n systems of equations. Symbolically these values may be written as $k_{ij} (i, j = 1, 2, \dots, n)$.

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[11] Two complex numbers are conjugate if they are of the form: $a+bi$ and $a-bi$.

Knowing the particular values of k and v, we can determine the solutions of the equation (7b) which are functions representing changes in time in the total products of particular branches: $X_1(t), X_2(t), \ldots, X_n(t)$, and each of these branches has n variants corresponding to different values of v.

Finally, the functions of total products satisfying equation (7b) and corresponding to the i-th branch of production, $X_i(t)$ can be written as follows:

$$k_{i1}e^{v_1 t}, k_{i2}e^{v_2 t}, \ldots, k_{in}e^{v_n t} \qquad (i = 1, 2, \ldots, n).$$

We know from the theory of differential equations that if certain functions constitute solutions of a system of linear equations, then their sum too, simple or weighted, is a solution of the given system of equations. It follows that the most general form of solutions of the system of equations (7b) is:[12]

$$X_i(t) = \sum_{j=1}^{n} h_j k_{ij} e^{v_j t}. \tag{11}$$

The coefficients h_j are arbitrary constants constituting the weights of the sums appearing on the right-hand side of formula (11).

Let us now analyse the economic meaning of the solutions of the system of equations (7b).

At the moment $t = 0$, the solutions (11) assume the following form:[13]

$$X_i(0) = \sum_{j=1}^{n} h_j k_{ij} \qquad (i = 1, 2, \ldots, n). \tag{11a}$$

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[12] These formulae would be more complex if equation (10) had multiple roots. A certain mathematical complication would arise but it would not influence the final conclusions. If, for instance, v_1 were the r-fold root then we would have:

$$X_i(t) = h_i(k_{i1}t^{r-1} + k_{i2}t^{r-2} + \ldots + k_{ir}) + \sum_{j=r+1}^{n} h_j k_{ij} e^{v_j t}\ .$$

Generally, the coefficients corresponding to a multiple root are polynomials of order t, by one less than the multiple of the root.

[13] The exponential function $e^{v_j t}$ for $t = 0$ equals 1.

This means that the weighted sum of the coefficients k_{ij} gives the value of the function of the total product $X_i(t)$ at the initial moment (zero). If the initial values of the total products of particular branches of production $X_1(0), X_2(0), \ldots, X_n(0)$ are known, then the constants $h_1, h_2, \ldots, h_n$ can be determined from the system of n equations (11a). In other words, the coefficients h_j are determined by *the initial conditions* of the process studied.

Let us now investigate the following question: what is the economic meaning of the quantities $v_1, v_2, \ldots, v_n$, and of the coefficients k_{ij} which are indirectly determined on their basis and what do they depend upon? The quantities v_i have been calculated from the characteristic equation (9) in which (besides v) there are the technical coefficients of production a_{ij} and the investment coefficients b_{ij}, expressing certain technical conditions, and the ratio of gross investment α_i. It follows that the quantities $v_i (i = 1, 2, \ldots, n)$ depend upon the technical conditions of production and the growth of production as well as upon economic decisions concerning the allocation of the total product to consumption and gross investment (compare formula (8) determining α_i). We express this by saying that the quantities $v_1, v_2, \ldots, v_n$ depend upon *the technical and economic structure of production.*

Moreover, if the roots $v_1, v_2, \ldots, v_n$ which constitute the solutions of the characteristic equation (9) are real, they represent the rate of growth of the total product of a given branch.

Let us consider, first, the case when the function of the total product $X_i(t)$ is expressed by only one component:

$$X_i(t) = h_1 k_{i1} e^{v_1 t}.$$

The velocity of growth of the total product $X_i(t)$ equals the derivative of this function:

$$\frac{dX_i(t)}{dt} = h_1 k_{i1} v_1 e^{v_1 t}.$$

But the rate of growth of the total product is the ratio of the velocity of growth of the total product to its size at a given moment:

$$\frac{\frac{dX_i(t)}{dt}}{X_i(t)} = \frac{h_1 k_{i1} v_1 e^{v_1 t}}{h_1 k_{i1} e^{v_1 t}} = v_1. \qquad (12)$$

In this case the coefficient v_1 gives the rate of growth of the total product.

In the general case, when a function solving the system of equations (7b) is the sum of a certain number of components, i.e. when:

$$X_i(t) = \sum_{j=1}^{n} h_j k_{ij} e^{v_j t},$$

we obtain

$$\frac{dX_i(t)}{dt} = h_1 k_{i1} v_1 e^{v_1 t} + h_2 k_{i2} v_2 e^{v_2 t} + \ldots + h_n k_{in} v_n e^{v_n t}.$$

The formula for the rate of growth is obtained by dividing both sides of this equation by $X_i(t)$:

$$\frac{\frac{dX_i(t)}{dt}}{X_i(t)} = \frac{h_1 k_{i1} v_1 e^{v_1 t} + h_2 k_{i2} v_2 e^{v_2 t} + \ldots + h_n k_{in} v_n e^{v_n t}}{h_1 k_{i1} e^{v_1 t} + h_2 k_{i2} e^{v_2 t} + \ldots + h_n k_{in} e^{v_n t}}. \tag{13}$$

The right-hand side of formula (13) can be interpreted as the weighted average of the roots $v_1, v_2, \ldots, v_n$. The roots $v_1, v_2, \ldots, v_n$ can be called *partial rates of growth* and the expression defined by formula (13) is called *the general rate of growth* of the total product of branch *i*.

Studying the general rate of growth of the total product of branch *i* we should bear in mind that the partial rates of growth may be more or less than zero or may equal zero; thus, the components of the sum determining the total product

$$X_i(t) = h_1 k_{i1} e^{v_1 t} + h_2 k_{i2} e^{v_2 t} + \ldots + h_n k_{in} e^{v_n t}$$

may be increasing, decreasing or constant functions.[14] It may happen, of course, that some of them are positive, some negative and some equal zero.

Let us consider, for the time being, some special cases. Let us assume that the partial rates of growth $v_1, v_2, \ldots, v_n$ differ in value, but are all positive. The graphs of the functions $e^{v_1 t}$,

[14] The exponential function $y = e^{vt}$ is increasing if $v > 0$, decreasing if $v < 0$ and constant and equal to 1 if $v = 0$.

$e^{v_2 t}, \ldots, e^{v_n t}$ constituting the variable factor of the particular components, determining the total product $X_i(t)$, are presented in Figure 6a. It can easily be seen from this graph that for small

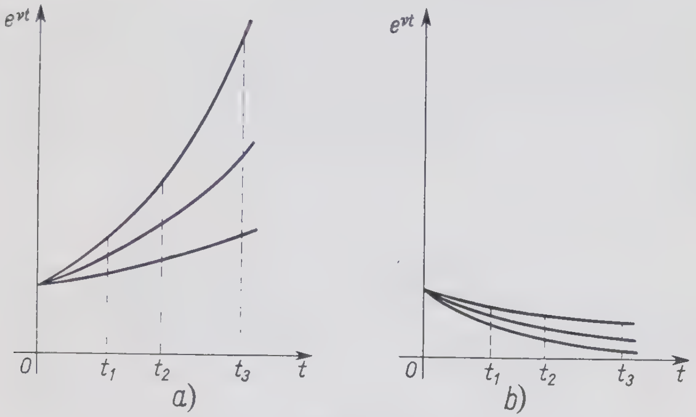

FIG. 6a, b.

values of t the differences are small between the values of the exponential functions $e^{v_1 t}, e^{v_2 t}, \ldots, e^{v_n t}$, and thus also between the particular components of the sum determining the total product $X_i(t)$. However, if it increases these differences become greater. In consequence, in the course of time, the component to which there corresponds the greatest rate of growth v, will exceed more and more the remaining components, so that they will cease to play a significant role.

In other words, the growth of the total product of branch i, considered as a function of time t:

$$X_i(t) = h_1 k_{i1} e^{v_1 t} + h_2 k_{i2} e^{v_2 t} + \ldots + h_n k_{in} e^{v_n t}$$

is composed of n trends and one of them—the one with the greatest rate of growth—is *dominant*.

The situation is similar when all partial rates of growth are negative. The graphs of the functions $e^{v_1 t}, e^{v_2 t}, \ldots, e^{v_n t}$ for each case are shown in Figure 6b. But here too one of the components of the sum

$$X_i(t) = \sum_{j=1}^{n} h_j k_{ij} e^{v_j t}$$

is dominant; it is the one for which the value of the rate of growth v is the greatest since the components with smaller rate of growth values tend faster to zero. The decline in the value of this component is the smallest as t increases.

When some partial rates of growth are positive and some negative, the dominant component is the increasing one with the corresponding highest rate of growth. An increasing trend, after a certain length of time, will always overcome a decreasing trend. In consequence, there will always appear some dominant trend: an increasing one, a decreasing one, or, in an exceptional case, a constant one.

Let us now consider a case in which some roots v are complex. We known that to each complex root v there corresponds another root, conjugate with it,[15] i.e. if there exists root v_j equal to $\alpha_j+i\beta_j$, then there also exists root v_{j+1} equal to $\alpha_j-i\beta_j$.

Let us consider one of the components of the sum $X_i(t) = \sum_{j=1}^{n} h_j k_{ij} e^{v_j t}$, for which v_j is a complex number.

Using Euler's formula,[16] we can transform this expression as follows:

$$h_j k_{ij} e^{v_j t} = h_j k_{ij} e^{(\alpha_j+i\beta_j)t} = h_j k_{ij} e^{\alpha_j t} e^{i\beta_j t}$$

$$= h_j k_{ij} e^{\alpha_j t} (\cos \beta_j t + i \sin \beta_j t).$$

Hence, we conclude that the component of the sum $X_i(t) = \sum_{j=1}^{n} h_j k_{ij} e^{v_j t}$, to which there corresponds the conjugate v_j, is a periodic function,[17] because the sum (difference) of periodic functions, which sine and cosine are, is also a periodic function.

[15] Here and in further formulae the letter "i" appearing as a factor denotes the imaginary number $i = \sqrt{-1}$; it should not be confused with the letter "i" used as a subscript for denoting the number of the branch of production.

[16] Euler's formulae: $e^{i\varphi} = \cos \varphi + i \sin \varphi$, $e^{-i\varphi} = \cos \varphi - i \sin \varphi$. In this case: $\varphi = \beta_j t$.

[17] If a time function is periodic, this means that after a certain length of time, called the period of the function, the pattern of the function repeats itself in an identical manner.

This component represents then a cycle in the development of total production $X_i(t)$.

The length of the period T_j of the periodic function considered, i.e. the length of the cycle, depends upon the argument of the trigonometric function, in this case—upon the quantity $\beta_j t$. To determine this period, let us note that the period of the sine function (as well as that of the cosine function) is 2π; we have then

$$\beta_j T_j = 2\pi,$$

hence

$$T_j = \frac{2\pi}{\beta_j}.$$

We see that β_j, in this case, is the factor which decides about the length of the cycle.

And what does α_j determine? To answer this question we shall consider three cases: $\alpha_j = 0$, $\alpha_j > 0$ and $\alpha_j < 0$.

(1) If $\alpha_j = 0$, then $e^{\alpha_j t} = 1$ and the cyclical component has the form $h_j k_{ij}(\cos \beta_j t + i \sin \beta_j t)$ and thus is the product of the constant quantity $h_j k_{ij}$ by the periodic function: $\cos \beta_j t + i \sin \beta_j t$. The cyclical component is then a function with a constant amplitude of fluctuations equal to $h_j k_{ij}$ (Fig. 7).

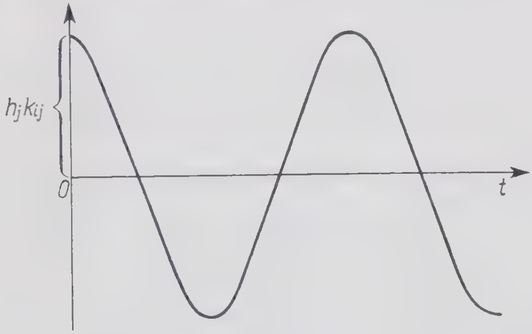

FIG. 7.

(2) If $\alpha_j > 0$, then the factor $e^{\alpha_j t}$ increases as t increases, and therefore also the amplitude of the cyclical factor, amounting to $h_j k_{ij} e^{\alpha_j t}$, increases sharply as an exponential function.[18] We can

[18] An exponential function (with a base greater than 1) is one of the fastest growing elementary functions.

see that in this case the fluctuations of the cyclical factor are increasing; because of a rapid rate of growth we say that they are of an explosive nature (Fig. 8).

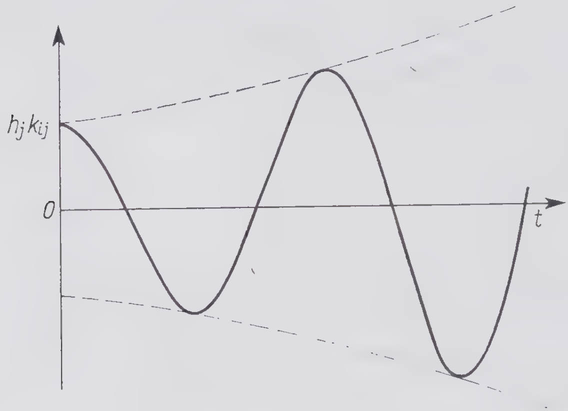

FIG. 8.

(3) If $\alpha_j < 0$, then, as is easy to show in a similar way, the fluctuations of the cyclical factor will be decreasing (dampened) because then the factor $e^{\alpha_j t}$ is a decreasing exponential function (Fig. 9).

It follows from these considerations that if the components of which the solution of the differential equation (7b): $X_i(t) = \sum_{j=1}^{n} h_j k_{ij} e^{v_j t}$ is composed have the exponents v_j complex, then the corresponding components are cyclical with the period of the cycle $T_j = \frac{2\pi}{\beta_j}$ and their fluctuations are constant, explosive or decreasing, depending upon the sign of α_j.

If there are more than one cyclical component with increasing fluctuations then, after a certain length of time t, one of them (the one in which α_j is the greatest), will become dominant and the remaining components will gradually lose in importance because, with the lapse of time, they become more and more dwarfed by the dominant cycle. Also dwarfed by the dominant cycle are the cycles with constant fluctuations and the cycles with decreasing fluctuations which gradually disappear.

We have stated that the particular solutions of the system of differential equations (7b) which are functions of the total production of particular branches, $X_i(t)$ $(i = 1, 2, \ldots, n)$ can be reduced to two kinds of components: increasing or decreasing trends (in exceptional cases—constant trends) and cycles with constant, increasing or decreasing fluctuations. It is found that among the trends one, as a rule, is dominant; similarly, among cycles there may appear a dominant cycle with increasing fluctuations.

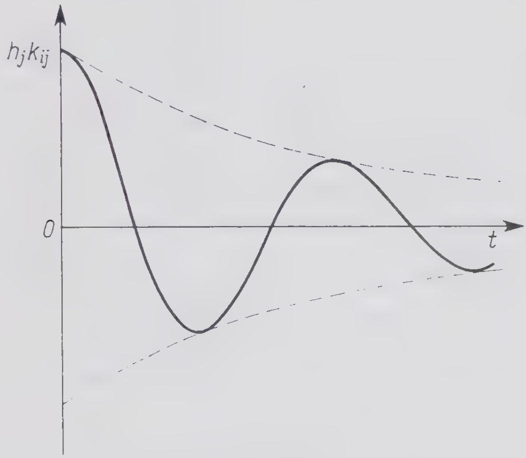

FIG. 9.

The theoretical considerations outlined above correspond to a specific reality. In a capitalist economy there exist cycles combined with a trend, usually a growing one.

The question arises: how many types of cycles are there in a capitalist economy and what are their lengths? In economic literature, business cycles of the duration of 8–10 years are usually discussed. But are there other cycles? American economists have pointed out that there appear short cycles lasting 3–4 years. The existence of such cycles can be established on the basis of graphs representing the economic development of the United States before the war. In European countries, however, short cycles have not been discovered.

There is an explanation of this fact. A long cycle is related to investments in fixed capital means, while in a short cycle changes in investments in working capital are dictated by changes in stocks. In the American economy stocks play a much more important part than in European countries where, in contrast to the United States, a substantial part of production is based on orders and there is no need for building up large stocks. It may be, however, that short cycles have not been discovered in European countries simply because statistical information was more scanty and less detailed.

Some economists[19] attempted to establish the existence of a third type of long-range cycles of 50–60 years duration. These economists used as a basis the years 1825–1842 which were years of long crises interrupted only by short periods of recovery. The years 1843–1873, on the other hand, were a period of a general economic boom and the recessions appearing during those years were short-lived. The years 1874–1896 were again a period of crises and depressions interrupted by short booms. The next period from 1897–1913 resembled the situation in the years 1843–1873.

The pattern of economic development in those long-range periods was said to consist of short business cycles of 8–10 years duration, superimposed upon a long-range cycle lasting 50 to 60 years.

If the theories on the existence of long-range and short-range (3–4 years) cycles were true, there should appear in our theoretical considerations three types of superimposed cycles.[20]

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[19] One of them was the Russian economist N. D. Kondratiev who published in 1925 the dissertation on "Bolshie tsikly konyunktury" *Voprosy konyunktury*, No. 1, 1925, pp. 28–79. Even earlier A. Spiethoff in the paper, Krisen in *Handwörterbuch der Staatswissenschaften*, 4th edition, Jena, 1923, drew attention to periods of intensive growth and periods of relative stagnation in the development of a capitalist economy, without interpreting these periods, however, as phases of cyclical development. The first to notice such periods were Parvus (A. L. Helphand), *Handelskrisis und Gewerkschaften*, Munich, 1901, p. 26, and Karl Kautsky, "Krisentheorien", *Die neue Zeit*, 1901/1902, vol. 2, pp. 136–143.

[20] This was J. A. Schumpeter's contention; he maintained that in a capitalist economy there appear three types of economic cycles: long-range cycles of 50–60 years, medium-range cycles lasting 9–10 years and short-range cycles of 3–4 years duration. See J. A. Schumpeter, *Business Cycles*, vol. 1, New York, London, 1939, p. 162–173.

The existence of long-range cycles has not been theoretically proved. Even though historical facts cited above are not subject to any serious reservations, they are not sufficient proof of the existence of long-range cycles. To prove this theory it would be necessary to show that there exists a causal relation between two consecutive phases of the cycle and nobody has succeeded in showing this. The pattern of economic development in the years 1825–1913 was the result of concrete historical facts of this period: rapid technical progress in transportation, industry and agriculture, rapid colonial expansion which created a steady demand for investment goods (shipbuilding, investments in overseas countries etc.). This led to consecutive periods of more or less intensive economic growth. For instance, the years from 1843 to 1873 were a period of rapid industrialization of Western European countries and of the extensive development of railways in Europe and North America. The years 1897–1913 were a period of large investments in overseas countries and of development of the electrical and chemical industries in Europe. These are concrete historical and economic phenomena and it is difficult to discern cycles in their pattern.

Let us deal, in greater detail, with the problem of trends and cycles. However, since it would be too complicated to consider this problem in general terms (to analyse the system of differential equations (7b) and their solution (11)), we shall confine ourselves to studying the case when the national economy is divided into two specialized divisions of production as in the Marxian schemes.

On the assumption that there are only two branches of production, the system of differential equations (7b) is reduced to two equations which, in a developed form, can be written as follows:

$$-\alpha_1 X_1(t)+a_{11}X_1(t)+a_{12}X_2(t)+b_{11}\frac{dX_1(t)}{dt}+b_{12}\frac{dX_2(t)}{dt}=0,$$

$$-\alpha_2 X_2(t)+a_{21}X_1(t)+a_{22}X_2(t)+b_{21}\frac{dX_1(t)}{dt}+b_{22}\frac{dX_2(t)}{dt}=0. \quad (14)$$

In this simplified case, the characteristic equation in the determinant form is as follows (compare with formula (9) for the general characteristic equation):

$$\begin{vmatrix} -\alpha_1+a_{11}+b_{11}v & a_{12}b_{12}v \\ a_{21}+b_{21}v & -\alpha_2+a_{22}+b_{22}v \end{vmatrix}=0. \quad (15)$$

Equation (15) is of the second degree with respect to v and therefore it has two real solutions or two complex and conjugate solutions.

The solutions of equations (14) may be written as follows:

$$\begin{cases} X_1(t) = h_1 k_{11} e^{v_1 t} + h_2 k_{12} e^{v_2 t}, \\ X_2(t) = h_1 k_{21} e^{v_1 t} + h_2 k_{22} e^{v_2 t}. \end{cases} \tag{16}$$

The numbers v_1 and v_2 are either both real or both complex and conjugate.

An interesting conclusion follows from this, namely if one of the components of the solutions (16) determines a certain cycle ($v_1 = \alpha + i\beta$), then the second component also determines a certain cycle ($v_2 = \alpha - i\beta$).

In consequence, there may appear in the solutions (16) either two trends (or one trend if $v_1 = v_2$), or two cycles. But the solutions (16) cannot consist of a cycle superimposed on a growing or declining trend.

Thus, if the economy is divided into two branches, we cannot obtain a model of economic development composed of a cycle and a trend, which in fact, is the pattern of the development of a capitalist economy. To obtain a realistic model of the development of a capitalist economy the national economy should be divided into at least three branches and a distinction should be made between, say, investments in fixed capital means and investments in working capital (stocks).

The division of the national economy into two branches, in a sense, conceals something, and hence we are not in a position to arrive at solutions of production balance-sheet equations that would produce simultaneously a cycle and a trend in the development of total products $X_1(t)$ and $X_2(t)$.

It may even happen that a cycle and a trend also appear simultaneously in the case when the national economy consists of two specialized divisions, if we introduce certain additional assumptions.

If the divisions of production are specialized, i.e. if $a_{21} = 0$ and $b_{21} = 0$, then there are neither reproduction nor investment flows from Division 2 to Division 1. However, we allow for the possibility that $a_{22} \neq 0$ and $b_{22} \neq 0$ which means that production in Division 2 may be used for the replacement of means of production of this division and for augmenting the stock of its products. Then, the second of the equations (14) will be reduced to:

$$-\alpha_2 X_2(t)+a_{22}X_2(t)+b_{22}\frac{dX_2(t)}{dt} = 0. \tag{17}$$

The characteristic equation will assume the following form:

$$\begin{vmatrix} -\alpha_1+a_{11}+b_{11}v & a_{12}+b_{12}v \\ 0 & -\alpha_2+a_{22}+b_{22}v \end{vmatrix} = 0.$$

or, after it is developed,

$$(-\alpha_1+a_{11}+b_{11}v)(-\alpha_2+a_{22}+b_{22}v) = 0. \tag{18}$$

Thus either

$$-\alpha_1+a_{11}+b_{11}v = 0,$$

or

$$-\alpha_2+a_{22}+b_{22}v = 0.$$

It follows that the characteristic equation (18) has two real solutions:

$$v_1 = \frac{\alpha_1-a_{11}}{b_{11}}, \qquad v_2 = \frac{\alpha_2-a_{22}}{b_{22}}. \tag{19}$$

The solutions of the system of equations (14), defined by formulae (16) for the case when $a_{21} = 0$ and $b_{21} = 0$ consist of two trends.

To determine when these trends are growing ones and when they are declining ones, it is necessary to check the signs of v_1 and v_2.

It follows from formulae (19) that $v_1 > 0$, i.e. the first trend is increasing when $\alpha_1-a_{11} > 0$, or $\alpha_1 > a_{11}$. Similarly, $v_2 > 0$, i.e. the second trend is increasing when $\alpha_2-a_{22} > 0$, or $\alpha_2 > a_{22}$.

Let us remember that α_1-a_{11} (and, similarly, α_2-a_{22}) is the difference between the gross rate of investment and the replacement coefficient, and b_{11} (or b_{22}) is the coefficient of investment outlays for the products retained in a given division of production. Therefore, quantity v_1 (or v_2) determining the rate of increase of the trend is a ratio of the rate of net investment in a given division to the coefficient of investment outlays from the products of this division.

The condition $\alpha_1 - a_1 > 0$, for which the first trend is a growing one, means that gross investment on the products of Division 1 is greater than the reproduction requirements, i.e. there is a net investment on the products of this division. The condition $\alpha_2 - a_{22} > 0$ can be interpreted in a similar way.

The trends will be decreasing if $\alpha_1 - a_{11} < 0$ and $\alpha_2 - a_{22} < 0$. These conditions mean that in these cases reproduction contracts since not all used up means of production can be replaced.

Finally, trends are constant when $\alpha_1 - a_{11} = 0$ and $\alpha_2 - a_{22} = 0$, i.e. $\alpha_1 = a_{11}$ and $\alpha_2 = a_{22}$.

These conclusions were evident *a priori* but the mathematical arguments outlined above will be useful for our further considerations.

Let us consider now what would happen if we assumed that $a_{22} = 0$, i.e. that production in Division 2 is not used for replacement in this division, but, as before, $b_{22} \neq 0$, i.e. accumulation takes place in Division 2 in the form of accumulation of stocks of products of this division. Then, the second solution (19) would be further simplified, namely:

$$v_2 = \frac{\alpha_2}{b_{22}}. \qquad (19a)$$

So far, our considerations which have led to solutions of balance-sheet equations without a cyclical pattern, concerned the situation prevailing in a socialist rather than in a capitalist economy because the rates of gross investment α_1 and α_2 in the analysed equations were assumed to be given in advance. We know, however, that in a capitalist economy the rates of gross investment α_1 and α_2 depend upon the profitability of particular branches of the economy and this, in turn, depends upon certain factors determined by the market mechanism.

If the rates of gross investment α_1 and α_2, appearing in the balance-sheet equations are considered as variable quantities determined by certain conditions influencing profitability there will appear in the solutions of the balance-sheet equations a cyclical component in addition to a trend component. Here, then, lies the basic difference between the course of reproduction and investment processes in a socialist economy and in a capitalist economy.

This idea may be expressed in yet another way. In a capitalist economy there exist certain relationships between investment and the profitability of production or the rate of profit. If these relationships are expressed by an additional equation, the solutions of balance-sheet equations will have both trend and cycle components. In a socialist economy such relationships between investment and profitability of production do not arise because the rates of gross investment α_1 and α_2 are determined by the economic plan.

It can be stated that the difference between reproduction and investment processes in a socialist economy and in a capitalist economy is that in a socialist economy the relationships between investment and profitability of production are set by the plan targets and not the spontaneous market mechanism. In a capitalist economy, on the other hand, there exist definite relationships between investment and profitability which may give rise to production cycles.

We shall attempt to define the relationship between investment and the rate of profit in a capitalist economy.

The equations of production inputs (formula (2) from Chapter 3) are:

$$X_i = x_{0i} + \sum_{j=1}^{n} x_{ji} + m_i \qquad (i = 1, 2, \ldots, n).$$

Introducing employment coefficients a_{0i} and production coefficients a_{ij} into this equation, it can be presented in the following form:

$$X_i = a_{0i} X_i + \sum_{j=1}^{n} a_{ji} X_i + m_i.$$

Thus, the surplus product, or profit[21] is:

$$m_i = X_i\Big((1 - a_{0i} - \sum_{j=1}^{n} a_{ji}\Big).$$

[21] Referring to a capitalist economy we call component m_i surplus value. We are not concerned here with the subtle question of distinction between surplus value and profit.

It follows that the profit per unit product[22] obtained in branch i is:

$$\pi_i = \frac{m_i}{X_i} = \left(1 - a_{0i} - \sum_{j=1}^{n} a_{ji}\right).$$

The simplest assumption that suggests itself in determining the relationship between investment and the rate of profit is that net investment on products of branch i, i.e.

$$X_i - x_i^{(0)} - \sum_{j=1}^{n} a_{ij} X_j = \alpha_i X_i - \sum_{j=1}^{n} a_{ij} X_j,$$

(where α_i denotes the rate of gross investment) is proportional to the rate of profit obtained from constant capital produced in this branch, i.e. $\frac{m_i}{K_i}$. K_i denotes the amount of products of a given branch that is included in the stock of constant capital of the national economy.

This assumption can be written as follows:

$$\alpha_i X_i - \sum_{j=1}^{n} a_{ij} X_j = \gamma_i \frac{m_i}{K_i} \qquad (i = 1, 2, \ldots, n), \tag{20}$$

where $\gamma_i > 0$ is the coefficient of proportionality.

Assumption (20) would result, however, in considerable computational complications.[23] We shall therefore adopt a simpler

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[22] The notion of "profit per unit product" $\pi_i = \frac{m_i}{X_i}$, i.e. profit calculated per unit product should not be confused with the "rate of profit" which is profit per unit of constant capital: $\frac{m_i}{K_i}$.

[23] Considering that $m_i = \pi_i X_i$ and $K_i = \sum_{j=1}^{n} b_{ij} X_j$, condition (20) can be written as follows

$$\alpha_i X_i - \sum_{j=1}^{n} a_{ij} X_j = \gamma_i \frac{\pi_i X_i}{\sum_{j=1}^{n} b_{ij} X_j},$$

assumption. We shall assume that net investment is a linear function of m_i and K_i.

$$\alpha_i X_i - \sum_{j=1}^{n} a_{ij} X_j = \gamma_i m_i - g_i K_i \quad (i = 1, 2, \ldots, n), \tag{21}$$

where $\gamma_i > 0$ and $g_i > 0$ denote the respective coefficients of proportionality.

The coefficient of proportionality γ_i can be called *the propensity coefficient of investments* in expectation of profits, and coefficient g_i can be called *the investment propensity coefficient for constant capital.*

Let us note that condition (21) is an approximation to condition (20) which is evident from the fact that investments increase when profit m_i increases, and they decline when the stock of constant capital K_i, consisting of the products of a given branch, increases.

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hence, after combining with balance-sheet equations, we obtain the equation:

$$-\gamma_i \frac{\pi_i X_i(t)}{\sum_{j=1}^{n} b_{ij} X_j} + \sum_{j=1}^{n} b_{ij} \frac{dX_j(t)}{dt} = 0.$$

The last equation can be presented in the following form:

$$-\gamma_i \pi_i X_i(t) + \sum_{j=1}^{n} b_{ij} X_j(t) \sum_{j=1}^{n} b_{ij} \frac{dX_j(t)}{dt} = 0,$$

or

$$-\gamma_i \pi_i X_i(t) + \sum_{j=1}^{n} \sum_{k=1}^{n} b_{ij} b_{ik} X_j(t) \frac{dX_k(t)}{dt} = 0.$$

$$(i = 1, 2, \ldots, n)$$

Substituting in the system of these equations $X_i = k_i e^{vt}$ (as a trial solution of the system of equations) we obtain:

$$-\gamma_i \pi_i k_i e^{vt} + \sum_{j=1}^{n} \sum_{k=1}^{n} b_{ij} b_{ik} k_i^2 v e^{2vt} = 0 \quad (i = 1, 2, \ldots, n)$$

This is a system of n quadratic equations with respect to k_i; it would not be easy to determine from it the values of k_i and v.

Condition (21) appearing in the process of reproduction and investment in a capitalist economy is an additional equation to production balance-sheet equations. In a socialist economy there is no such additional equation for the process of reproduction and investment.

Considering that:

(1) from the previous assumption the amount of the product of branch i used in branch j, as constant capital, is proportional to production in branch j, it follows that $K_{ij} = b_{ij}X_j$, where b_{ij} are the corresponding investment coefficients.

Therefore:

$$K_i = \sum_{j=1}^{n} K_{ij} = \sum_{j=1}^{n} b_{ij}X_j.$$

(2) $m_i = \pi_i X_i$, where π_i denotes profit per unit product in the i-th branch, we can transform equation (21) in the following way:

$$\alpha_i X_i - \sum_{j=1}^{n} a_{ij}X_j = \gamma_i \pi_i X_i - g_i \sum_{j=1}^{n} b_{ij}X_j \qquad (i = 1, 2, \ldots, n). \quad (22)$$

Combining these additional equations with the balance-sheet equations:

$$-\alpha_i X_i(t) + \sum_{j=1}^{n} a_{ij}X_j + \sum_{j=1}^{n} b_{ij}\frac{dX_j(t)}{dt} = 0 \qquad (i = 1, 2, \ldots, n),$$

we arrive at the new equations:

$$-\gamma_i \pi_i X_i(t) + g_i \sum_{j=1}^{n} b_{ij}X_j(t) + \sum_{j=1}^{n} b_{ij}\frac{dX_j(t)}{dt} = 0$$

$$(i = 1, 2, \ldots, n), \qquad (23)$$

which allow for the additional condition, making the rates of investment α_i dependent upon the rate of profit, and thus correspond to the conditions prevailing in a capitalist economy.

We shall introduce one more additional assumption.[24] We

[24] This additional condition constitutes a basis for the assumption on which Kalecki's model of business cycles is based. See lectures by O. Lange, *Teoria rozwoju gospodarczego* (*Theory of Economic Development*), Part II, Warsaw University Publishing Department, 1958.

assume that the volume of investment at time t depends upon the profitability of the respective branches of the economy at time $t-\theta$, i.e. at a moment earlier by period θ which we shall call *the period of the realization of investment.*[25]

From this assumption and from the method of deriving equation (23), it follows that if investments $\frac{dX_j}{dt}$ correspond to moment t, then the rate of profit determining these investments corresponds to moment $t-\theta$. Thus, equation (23) will assume the following form:[26]

$$-\gamma_i \pi_i X_i(t-\theta)+g_i \sum_{j=1}^{n} b_{ij} X_j(t-\theta)+\sum_{j=1}^{N} b_{ij} \frac{dX_j(t)}{dt}=0$$

$$(i=1, 2, \ldots, n). \qquad \text{(23a)}$$

If the period for the realization of an investment θ is taken as a unit of time ($\theta = 1$), then

$$-\gamma_i \pi_i X_i(t-1)+g_i \sum_{j=1}^{n} b_{ij} X_j(t-1)+\sum_{j=1}^{n} b_{ij} \frac{dX_j(t)}{dt}=0$$

$$(i=1, 2, \ldots, n). \qquad \text{(23b)}$$

Equation (23b) differs from normal differential equations in that the magnitudes of unknown functions $X_j(t)$ $(j=1, 2, \ldots, n)$, appear at different moments, namely at moment t and moment $t-1$. Equations of this kind are called differential-difference equations; a system of such equations is solved in the same way as a system of differential equations.

Let us assume that the solutions of the system of equations (23b) are in the form $X_j(t)=k_j e^{vt}$, and let us substitute these values in the system of equations (23b). We obtain:

$$-\gamma_i \pi_i k_i e^{v(t-1)}+g_i \sum_{j=1}^{n} b_{ij} k_j e^{v(t-1)}+\sum_{j=1}^{n} b_{ij} k_j v e^{vt}=0. \qquad \text{(23c)}$$

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[25] This assumption is indeed realistic because some time must elapse between the moment of making an investment decision and its realization.

[26] In this way, equation (23b) contains variables corresponding to different moments of time. In other words, there is a time-lag between the quantities of total products and investments.

After dividing[27] by $e^{v(t-1)} \neq 0$ we arrive at the following system of algebraic equations homogeneous with respect to the unknowns $k_j (j = 1, 2, \ldots, n)$

$$-\gamma_i \pi_i k_i + g_i \sum_{j=1}^{n} b_{ij} k_j + \sum_{j=1}^{n} b_{ij} k_j v e^v = 0 \quad (i = 1, 2, \ldots, n). \quad (24)$$

The characteristic equation of the system of equations (23c) in the determinant form can be written as follows:

$$\begin{vmatrix} b_{11}(g_1+ve^v)-\gamma_1\pi_1 & b_{12}(g_1+ve^v) & \ldots b_{1n}(g_1+ve^v) \\ b_{21}(g_2+ve^v) & b_{22}(g_2+ve^v)-\gamma_2\pi_2 & \ldots b_{2n}(g_2+ve^v) \\ \ldots & \ldots & \ldots \\ b_{n1}(g_n+ve^v) & b_{n2}(g_n+ve^v) & \ldots b_{nn}(g_n+ve^v)-\gamma_n\pi_n \end{vmatrix} = 0. \quad (25)$$

The further determination of the general solutions of the system of equations (23b) and the analysis of these solutions is analogous to the previously discussed solution of the system of equations (7b). We shall therefore not concern ourselves with a general solution of the system of equations (23b) but we shall investigate the most interesting case of an economy divided into two specialized divisions: the production of means of production and the production of means of consumption. In the case when $n = 2$, the left-hand side of the characteristic equation (25) is reduced to the determinant of the second degree:

$$\begin{vmatrix} b_{11}(g_1+ve^v)-\gamma_1\pi_1 & b_{12}(g_1+ve^v) \\ b_{21}(g_2+ve^v) & b_{22}(g_2+ve^v)-\gamma_2\pi_2 \end{vmatrix} = 0. \quad (26)$$

Since in the case of specialized divisions $b_{21} = 0$ the characteristic equation (26) assumes the form:

$$[b_{11}(g_1+ve^v)-\gamma_1\pi_1][b_{22}(g_2+ve^v)-\gamma_2\pi_2] = 0. \quad (26a)$$

Equation (26a) holds if either the first or the second factor, appearing on the left-hand side of the equation, equals zero. There are, then, two solutions, v_1 and v_2, of the characteristic equation (26a), determined by the conditions:

$$v_1 e^{v_1} = \frac{\gamma_1 \pi_1}{b_{11}} - g_1, \quad (27a)$$

[27] Let us note that $e^{vt} = e^v \cdot e^{v(t-1)}$.

and

$$v_2 e^{v_2} = \frac{\gamma_2 \pi_2}{b_{22}} - g_2. \tag{27b}$$

If the roots of the characteristic equation (26a) are real then, as we know from our previous general considerations, the functions determining the sizes of the total products, in this particular case, functions $X_1(t)$ and $X_2(t)$ contain the components of an increasing, decreasing or constant trend. If, however, the roots of the characteristic equation are complex, then the function $X_1(t)$ and $X_2(t)$ contain a cyclical component.

Whether equations (27a) and (27b) have real or complex roots depends upon the values of b_{11}, b_{22}, π_1, π_2, g_1, g_2, γ_1 and γ_2.

It is possible to determine real roots v_1 and v_2 from equations (27) by a graphical method. Considering v_1 as an independent variable, we draw a graph of the function $y = v_1 e^{v_1}$ appearing on the left-hand side of equation (27a), and a graph of the constant function $y = \frac{\gamma_1 \pi_1}{b_{11}} - g_1$, appearing on the right-hand side of the equation. If there are points of intersection of the curves representing these functions, then the abscissae of these points determine the real roots of equation (27a). If, however, there are no points of intersection of the curves of functions: $y = v_1 e^{v_1}$ and $y = \frac{\gamma_1 \pi_1}{b_{11}} - g_1$, then the roots of equation (27a) are complex.

In a similar way, we determine the roots of equation (27b).

Let us draw a graph of functions: $y = v_1 e^{v_1}$ and $y = \frac{\gamma_1 \pi_1}{b_{11}} - g_1$.

The graph of the second of these functions which does not contain variable v_1, is a straight, horizontal line intersecting the y-axis at the point whose ordinate equals $\frac{\gamma_1 \pi_1}{b_{11}} - g_1$ (Fig. 10). To draw a graph of function $y = v_1 e^{v_1}$, let us note that this function is continuous, and that for $v_1 = 0$ it assumes the value $y = 0$, and for $v_1 = 1$ it assumes the value $y = e$. Moreover, this function reaches its extreme value (minimum) for $v_1 = -1$. Indeed, $y' = e^{v_1} + v_1 e^{v_1} = e^{v_1}(1 + v_1)$. Hence the extreme value obtains for $v_1 = -1$, since then $y' = 0$ and the extreme value equals: $y_{\min} = -\frac{1}{e} \approx -0.368$.

It is easy to check further that when $v_1 \to -\infty$, $y \to 0$[28] (the axis of abscissae is then an asymptote of the curve $y = v_1 e^{v_1}$), and when $v_1 \to +\infty$, $y \to +\infty$.

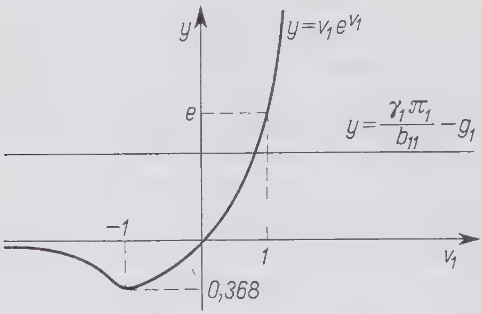

Fig. 10.

This discussion indicates the position and the shape of the curve representing the function $y = v_1 e^{v_1}$. This is shown in Fig. 10. Moreover, from Fig. 10 we infer directly that:

(1) if

$$\frac{\gamma_1 \pi_1}{b_{11}} - g_1 \geqslant 0,$$

then equation (27a) has one real root which is positive when the inequality > 0 holds, and equals zero when the equality $= 0$ obtains.

(2) if

$$0 > \frac{\gamma_1 \pi_1}{b_{11}} - g_1 \geqslant -0.368,$$

then equation (27a) has two negative real roots, in a special case one negative double root when

$$\frac{\gamma_1 \pi_1}{b_{11}} - g_1 = -0.368;$$

[28] This follows from the fact that the limit of product $v_1 e^{v_1}$ depends upon factor e^{v_1} which sharply tends to zero when $v_1 \to -\infty$. Factor e^{v_1} dominates over the linear factor v_1. The limit of product $v_1 e^{v_1}$ for $v_1 \to -\infty$ can also be determined by the de l'Hospital rule because it is an indeterminate expression of the type $\infty \cdot 0$.

(3) if, however

$$\frac{\gamma_1\pi_1}{b_{11}}-g_1 < -0.368,$$

then there are no real roots.

In a similar way, it is possible to determine conditions for the existence of real roots for equation (27b).

Depending upon real and complex roots in equations (27a) and (27b), the corresponding functions $X_1(t)$ and $X_2(t)$, representing increments in the total products of Division 1 and Division 2, may contain components determining a growing trend when $v_1 > 0$, a declining trend when $v_1 < 0$ and a constant trend when $v_1 = 0$, $(i = 1, 2)$.

Let us now consider possible complex roots of the equations (27a) and (27b). We shall discuss equation (27a); the results will be applicable to equation (27b). After substituting $v_1 = \alpha+i\beta$ in equation (27a) we obtain:

$$(\alpha+i\beta)e^{\alpha+i\beta} = \frac{\gamma_1\pi_1}{b_{11}}-g_1,$$

or

$$(\alpha+i\beta)e^{\alpha} = \left(\frac{\gamma_1\pi_1}{b_{11}}-g_1\right)e^{-i\beta}.$$

Using Euler's Theorem $e^{-i\beta} = \cos\beta - i\sin\beta$, we can write it in the following form

$$(\alpha+i\beta)e^{\alpha} = \left(\frac{\gamma_1\pi_1}{b_{11}}-g_1\right)(\cos\beta - i\sin\beta).$$

The real part on the left-hand side of the equation must equal the real part on the right-hand side and the imaginary part on the left-hand side must equal the imaginary part on the right-hand side. It follows that

$$\alpha e^{\alpha} = \left(\frac{\gamma_1\pi_1}{b_{11}}-g_1\right)\cos\beta, \tag{28a}$$

and

$$\beta e^{\alpha} = -\left(\frac{\gamma_1\pi_1}{b_{11}}-g_1\right)\sin\beta. \tag{28b}$$

Dividing the second equation by the first and considering that $e^\alpha \neq 0$, we obtain

$$\frac{\beta}{\alpha} = -\tan\beta,$$

or[29]

$$\tan\beta = -\frac{\beta}{\alpha}. \tag{29}$$

The real part α and coefficient β in the imaginary part of complex roots must satisfy equation (29). Since both α and β are real, equation (29) can be solved graphically. Shown in Figure 11 is the graph of the function $\tan\beta$ in the interval $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ and the straight line $y = -\frac{\beta}{\alpha}$. The points of intersection of this straight line with the graph of function $\tan\beta$ determine the roots of equation (29).

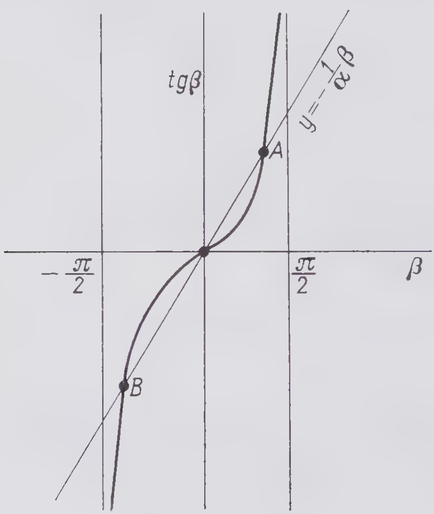

FIG. 11.

[29] We also assume that $\frac{\gamma_1 \pi_1}{b_{11}} - g_1 \neq 0$. In the case when $\frac{\gamma_1 \pi_1}{b_{11}} - g_1 = 0$ we obtain the result (29) by moving to the limit on the basis of de L'Hospital's Theorem.

The straight line intersects the graph of function $\tan\beta$ at the origin of the system of coordinates. This intersection produces the result $\beta = 0$, i.e. it corresponds to a real root if it exists. Complex roots correspond to the point of intersection for which $\beta \neq 0$. The existence of such intersections depends upon the slope of the straight line which is $-\frac{1}{\alpha}$. For the straight line to intersect the graph of function $\tan\beta$ at other points than the origin of the system of coordinates its slope must be positive i.e., $-\frac{1}{\alpha} > 0$, or $\alpha < 0$. Moreover, the slope of the straight line must be greater than the slope of the graph of the function in the origin of the system, where the graph of the function $\tan\beta$ has the slope 1, and therefore[30] $-\frac{1}{\alpha} > 1$, or, in consequence, $-1 < \alpha < 0$. If this condition is satisfied, the straight line intersects the graph of function $\tan\beta$ at two symmetrical points (points A and B in Figure 11) and the complex roots constitute a conjugate pair $\alpha + i\beta$ and $\alpha - i\beta$.

As we have seen, the condition of the existence of complex roots of a characteristic equation is that the inequality $-1 < \alpha < 0$, is satisfied. This means that the cycle is dampened, and the left-hand side of this inequality imposes a certain limit upon the degree of dampening the cycle. In consequence, however, in the course of time, the cycle fades away. To bring about new fluctuations of the cycle there must arise external disturbances which would impair the equilibrium achieved on the trend line. In a capitalist economy such disturbances appear all the time in the form of technical and organizational progress, the availability of new raw materials, sudden changes in demand (for instance as a result of armaments or other public investments). They cause the diminishing business cycle to be always stimulated and the cyclical pattern to be a permanent feature of development of a capitalist economy.

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[30] Since $\frac{d\tan\beta}{d\beta} = 1 + \tan^2\beta$ holds, therefore at the origin of the system where $\beta = 0$, we have $\frac{d\tan\beta}{d\beta} = 1$.

Figure 11 shows that there is a certain relationship between value α, which determines changes in the amplitude of fluctuations and value β, which determines the length of their period. Value β, determined by the point of intersection A, is contained within the interval $\left(0, \frac{\pi}{2}\right)$ and increases as the slope coefficient of the straight line $y = -\frac{1}{\alpha}\beta$, increases, i.e. as the absolute value of α decreases. The period of fluctuations, i.e. the length of the cycle, is[31] $T = \frac{2\pi}{\beta}$. The greater the absolute value α, i.e. the more dampened the cycle (the faster it fades away), the smaller value β and the longer the period T. It turns out that strongly dampened cycles are shorter and less dampened cycles are longer. Since the upper limit of the possible values of β is $\frac{\pi}{2}$ it turns out that the length of the cycle cannot be shorter than 4 units of time (we have assumed that the unit of time is the period of realization of investment θ).

Function $\tan\beta$ is periodic and its length is π. Equation (29) therefore, has roots also outside the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If the equation is satisfied by a specific value β_0 contained in this interval, then it is also satisfied by the values $\beta_0 \pm k\pi$, where $k = 1, 2, \ldots$ since $\tan\beta = \tan(\beta + k\pi)$ for integral values of k. In addition to root β_0 there are also roots $\beta_0 + \pi$, $\beta_0 + 2\pi$ etc. To these roots there correspond additional fluctuations with periods $T = \frac{2\pi}{\beta_0 + \pi}$, $T = \frac{2\pi}{\beta_0 + 2\pi}$, $T = \frac{2\pi}{\beta_0 + 3\pi}$ etc., i.e. fluctuations that become shorter and shorter. Since $\beta_0 > 0$, then the periods of these fluctuations are $T < 2$, $T < 1$, $T < 3/4$ etc. Except for the first one, all the additional fluctuations last for less than one unit of time and, therefore, in practice cannot manifest themselves and may be disre-

[31] Denoting by T period of fluctuations we have $\beta T = 2\pi$ and hence $T = \frac{2\pi}{\beta}$.

garded. We should consider only the first of the additional fluctuations which may manifest itself. Since, as we know, $\beta_0 < \frac{\pi}{2}$ we find that the period of this fluctuation is contained within the limits $4/3 < T < 2$. In addition to the cycle lasting at least 4 units of time another cycle (of the length 4/3 to 2 units of time) is also possible. This cycle also expires in the course of time since value α is the same for all fluctuations.

Finally, we shall investigate under what conditions there may appear both real and complex roots of equations (27a) and (27b), i.e. under what conditions there may appear both cycles and trends. Let us note that complex roots in addition to equation (29) must also satisfy the equations (28a) and (28b). Since equation (29) is derived from the two latter equations it suffices that they satisfy one of them, and then the second will also be satisfied. Let us consider equation (28a). The equation (29) is satisfied if $-1 < \alpha < 0$, and therefore the left-hand side of equation (28a) must satisfy the condition (see Figure 10 substituting α for v_1).

$$-\frac{1}{e} < \alpha e^{\alpha} < 0,$$

or

$$-0.368 < \alpha e^{\alpha} < 0.$$

The same inequalities must be satisfied by the right hand side of equation (28a), i.e.

$$-0.368 < \left(\frac{\gamma_1 \pi_1}{b_{11}} - g_1\right) \cos \beta_0 < 0, \tag{30}$$

where β_0 is determined by equation (29).

In the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ inequality $0 < \cos \beta < 1$ holds. Therefore inequality (30) can be satisfied for $-\frac{\pi}{2} < \beta_0 < \frac{2}{\pi}$ only if

$$-0.368 < \frac{\gamma_1 \pi_1}{b_{11}} - g_1 < 0,$$

i.e. in the case when there are two negative real roots. In this case there exist two declining trends which tend to zero. As a result there remain two expiring cyclical fluctuations around a specific value of the average.

There is a growing trend (the positive real root) if

$$\frac{\gamma_1 \pi_1}{b_{11}} - g_1 > 0.$$

Because of (30) this is possible only if $\cos\beta < 0$. This takes place if β equals $\beta_0+\pi$, $\beta_0+3\pi$, $\beta_0+5\pi$ etc. Except for the first one, these intervals, as we have seen, correspond to cycles with a period shorter than one unit of time (assumed to equal θ), which do not manifest themselves. The first of these intervals corresponds to the cycle of length $4/3 < T < 2$. It turns out that a growing trend may appear only together with a cycle shorter than two units of time.

As a result we can see that declining trends are combined with a diminishing cycle of the length of at least four units of time and a growing trend can be combined with a diminishing cycle of the length of less than two time units. To explain this result let us stop and think of the causes of business cycles and development trends in a capitalist economy. It follows from the conditions determining equations $X_1(t)$ and $X_2(t)$ that the nature of these solutions depends upon the value of the coefficients of sensitivity of investments to fixed capital g_1 and g_2, and upon the value of the coefficients of sensitivity of investments to profits, i.e. γ_1 and γ_2. The cycle is a result of the sensitivity of investments to fixed capital whose increase causes a decline in the rate of profit and a decline of investments. An increase in fixed capital hampers an increase in investments and stops an increase in production. Similarly, in the declining phase of the cycle, a decrease in fixed capital stimulates investments and results in an increase in production.

In a socialist economy the hampering influence of fixed capital (means of production) on investments is removed and therefore the cyclical pattern of the production process is also removed. Investments and production grow in a planned way.

The question arises whether the cycle returns to the starting level of fixed capital and production or whether it ends at a level higher or lower than the starting point and at what level a new cycle starts? This depends upon the sign of the difference

$$\frac{\gamma_1 \pi_1}{b_{11}} - g_1,$$

or

$$\frac{\gamma_2 \pi_2}{b_{22}} - g_2 .$$

As we have seen, if this difference is positive the characteristic equation has a positive real root and therefore there is a growing trend which is imposed upon the business cycle. In this case, the sensitivity of investments to profit exceeds the sensitivity of investments to fixed capital and the stimulus to expand is stronger than the stimulus to contract. In consequence, the cycle ends at a higher level of fixed capital and production than that at which it began and there is a growing trend. In other words, the process of expanded reproduction takes place through the fluctuations of business cycle. But for the very reason that the stimulus to contract is weak in relation to the stimulus to expand, after a very short period of contraction in investment and of decrease in fixed capital, the factors stimulating investments and production become dominant again. In consequence the business cycle is short.

If, however, the difference mentioned above is negative (but greater than -0.368) there appear declining trends. This results from the fact that the sensitivity of investments to fixed capital exceeds the sensitivity of investments to profit and the stocks of fixed capital and production contract from one cycle to another (the process of contracted reproduction takes place) and they finally stabilize at the level of the average value of these quantities in the business cycle. Under these circumstances, because of a relatively high sensitivity of investments to fixed capital which has a contracting effect, the declining phase of the cycle is long and the stock of fixed capital contracts so much that again the factors stimulating growth become dominant and they express themselves in the sensitivity of investments to profits. Under these conditions the business cycle is long.

On the basis of the above considerations we cannot determine, however, the real periods of the cycles mentioned above since we do not know the length of period θ. Only a statistical investigation could give a satisfactory answer to this question.

M. Kalecki assumes on the basis of some statistical data that $\theta = 1$ year.[32] It seems, however, that the length of time that

[32] See M. Kalecki: *Teoria dynamiki gospodarczej* (*The Theory of Economic Dynamics*, in Polish), Warsaw, 1958, p. 181 and ff.

elapses between making an investment decision and its realization is greater. If it is assumed that $\theta = 2$ years, then for constant cycles we would obtain a period not exceeding 8 years. Assuming the average length of θ between 1 and 2 years, a business cycle should last from 4 to over 8 years. This is a typical cycle observed in a capitalist economy. With a growing trend the cycle should be shorter: less than 2 years for $\theta = 1$ and less than 4 years for $\theta = 2$, on the average somewhere between 2 and 4 years at the most.

In this connection, the following comments are relevant:

First of all, a model of the development of a capitalist economy can be considered as realistic only if the length of the cycle derived from it corresponds to the real length of a business cycle, i.e. 7–9 years. Models in which there appear too long or too short business cycles in comparison with the 7–9 years cycle should be rejected as unrealistic.

Secondly, our analysis concerning the length of business cycles was only a rough approximation to real economic conditions, for we have assumed that the national economy was divided into two divisions. A more extensive analysis would entail dividing the economy into a larger number of branches and especially drawing a clear distinction between investments in fixed capital means with a relatively long period of depreciation and investments in working capital.

Then, probably, in addition to a business cycle of 7–9 years duration there would appear also a shorter cycle of 3–4 years as has been discovered in the United States. A division of the economy into a larger number of branches would, perhaps, make it possible to discover the existence of some other cycles, typical, say, to construction, agriculture, etc.

Moreover, studies on the cyclical nature of the development of the national economy should also take into account the phenomenon known as "echo".

It is well known that machinery and other equipment used as means of production are not uniformly distributed in time as far as the moment of their purchase is concerned. At times of prosperity, as a rule, all entrepreneurs invest and then the time comes when there is no substantial demand for means of production. In turn, after a period of time corresponding to the average period of depreciation of invested equipment it becomes necessary to replace a large portion of it within a short space of time. This situation may repeat itself several times at more or less the same intervals. Such a periodic repetition of concentrated replacement is in itself a cyclical phenomenon.

While in a socialist economy the phenomenon of business cycles does not appear because large-scale investment propensity for constant capital has been eliminated, nevertheless a cyclical phenomenon of an "echo", or a cumulative replacement of means of production may appear also in a socialist economy. The beginning of an echo may be stimulated by, say, a process of socialist industrialization which is characterized by building numerous industrial establishments in a relatively short space of time. These establishments, after a certain number of years, may have to be more or less simultaneously replaced because of physical wear and tear or obsolescence. We shall deal with this problem in the next chapter.

Chapter 5

DEPRECIATION AND REPLACEMENT PROBLEMS

In our considerations we have not distinguished so far between depreciation and replacement. We have tacitly assumed that the rate of depreciation equals the value of worn out constant capital (fixed capital means of production) and amounts to $\frac{1}{T}$ part of its value, where T denotes the average lifespan of the means of production and $\frac{1}{T}$ is called *the rate of depreciation.* This is so, however, only under conditions of simple reproduction. In the case of expanded reproduction, we shall see that the situation is more complicated.

Let us assume that an enterprise has 100 machines at its disposal and that their lifespan is $T = 20$ years. Let us also assume, for the sake of simplicity, that all the machines have been installed at the same time. Over the period of 20 years none of these machines will be withdrawn from production, but at the end of that period all the machines will have to be replaced simultaneously.

However, to make such a replacement possible, the entrepreneur puts aside each year a portion of the value of production in an amount equal to the *rate of depreciation* which for each machine amounts to 1/20 of its value. The aggregate annual amount of depreciation for 100 machines corresponds, in this case, to the value of 5 machines ($100 \cdot 1/20 = 5$). After 20 years, the enterprise will be able to buy exactly 100 new machines, and thus the stock of these means of production will not change.

Let us assume further that we are faced with a process of expanded reproduction and that the number of machines increases by, say, 10 per cent each year. Thus, the original number of 100 machines increases gradually every year in the following way:

1st year	—	100
2nd ,,	—	110
3rd ,,	—	121
.... etc.		

If the rate of depreciation is, as in the previous example, $1/20 = 5$ per cent, then the amount of depreciation written off annually will be:

in the 1st year	—	the value of	5	machines
,, 2nd ,,	—	,, ,,	5.5	,,
,, 3rd ,,	—	,, ,,	6.05	,,
.... etc.				

After 20 years the enterprise will have to replace 20 machines but, as can easily be seen, it will have put aside more than ample funds for the purchase of the same number of machines as before. The depreciation reserve (fund) is, then, larger than needed for the replacement of the used up means of production. The surplus obtained in this way can be used for, e.g. net investment, i.e. for increasing the stock of means of production.

It follows that depreciation and replacement are two different notions. Depreciation is a reserve obtained by "writing off" a portion of the value of the product for the purpose of replacement. Replacement is a part of the value of the product that must actually be used at a given moment (or period) for replacing the used up means of production. Depending upon the method of writing off the depreciation reserve may or may not coincide, at a given moment (period), with the actual replacement needs.

There are various formulae and systems in use for arriving at the amount of depreciation. The simplest is *linear depreciation* which consists in writing off every year the same portion of the value of constant capital means. It follows from the examples given above that in the case of simple reproduction, depreciation determined in this way corresponds to replacement requirements; but in the case of expanded reproduction the depreciation reserve is higher than the replacement requirements.

In connection with these considerations two questions arise:

(1) What is the relationship between the depreciation reserve and the replacement requirements when the method of linear depreciation is used?

(2) Is it possible to arrive at a formula for calculating depreciation whereby the depreciation reserve would correspond exactly to the replacement requirements?

It is found that the answer to the second question is affirmative and that such a formula can be produced. Nevertheless, in practice linear depreciation is almost always used. This is so both in the United States and in socialist countries although in the literature on this subject the method of linear depreciation is often criticized. It is argued that the formula affording a depreciation reserve balancing exactly replacement requirements should be used instead.

However, the realization of these recommendations in practice is not easy. First of all, such a depreciation formula is complicated and, secondly, in all countries there are, in force, taxation or accounting regulations which determine the principles and the amount of depreciation.

Moreover, the system of linear depreciation is more advantageous to capitalist enterprises since for growing enterprises the burden of taxation on profits is smaller and they thus have available additional funds for investment purposes.[1]

The problems of depreciation and replacement have been the subject of dicsussion for a long time. Marx, in volume 2 of *Capital*, pointed out that the depreciation reserve may be a source of accumulation and investment.[2] The American economist, E. D. Domar, applied a mathematical method of analysis to this problem.[3]

We shall now try to determine in great detail replacement requirements under conditions of expanded reproduction, and to analyse the relationship between the reserve necessary for replace-

[1] In some countries (e.g. in England) tax regulations permit the use of accelerated depreciation which reduces even further the tax on profit.

[2] See *Capital*, vol. 2.

[3] See E. D. Domar, *Essays in the Theory of Economic Growth*, New York, 1957, Chapter 7. The results obtained in this field by E. D. Domar and by other economists are described in Chapter 5 of the book by W. Lissowski, *Problem zużycia ekonomicznego środków pracy* (*The Problem of the Economic Wear and Tear of the Means of Labour*, in Polish), Warsaw 1958.

ment and the depreciation reserve built up by the method of linear depreciation.

To simplify, let us assume that all capital means of production have been brought into the process of production simultaneously and that they have the same lifespan. We shall denote in our further considerations this lifespan by the letter m (in the notation used above $m = T$) as is accepted in the literature on the subject. If the value of the stock of constant capital means of production is denoted[4] by K, then the amount of depreciation written off by the linear method is determined by formula: $A = \frac{K}{m}$, or $A = aK$ where $a = \frac{1}{m}$ is the rate of depreciation.

Let us denote by R the value of capital objects subject to replacement. In economic literature they are known as *restitution investments.*

The quantities K, A and R change and therefore are functions of time t. The value of capital objects at the initial moment (or period) will be denoted by symbol K_0 and the quantities K, A and R at moment (period) t—by symbols K_t, A_t and R_t. For the time being, we shall analyse the values of K_t, A_t and R_t in finite periods, e.g. in particular years.

If we assume that the rate of growth of additional gross investments (i.e. capital) in the starting period is K_0, then the value of the capital object in period t is:[5]

$$K_t = K_0 + K_0(1+r) + K_0(1+r)^2 + \ldots + K_0(1+r)^{t-1},$$

or

$$K_t = K_0[1+(1+r)+(1+r)^2+ \ldots +(1+r)^{t-1}]. \qquad (1)$$

The right-hand side of formula (1) is a geometric progression whose quotient equals $1+r$ and therefore

$$K_t = K_0 \frac{(1+r)^t - 1}{(1+r)-1},$$

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[4] Quantity K is, then fixed "capital" in the sense used in an analysis of a capitalist economy. A fixed capital means of production is usually called "capital object".

[5] It follows from the assumption that the growth of the value of a capital object starts from zero, i.e. the value of the capital object in the first period corresponds to the amount of investment and is K_0, in the second period $K_0 + K_0(1+r)$, in the third $K_0 + K_0(1+r) + K_0(1+r)^2$, etc.

or

$$K_t = K_0 \frac{(1+r)^t - 1}{r}. \tag{1a}$$

Formula (1a) expresses the relationship between the initial value of capital object K_0, the rate of growth of additional gross investments r and the value of the capital object in period t.

We shall now determine two relations: (1) the relation between depreciation A_t and gross investment B_t, and (2), the relation between replacement R_t and depreciation A_t.

The amount of gross investment in period t is $B_t = K_0(1+r)^t$ and depreciation $A_t = \frac{K_t}{m}$; therefore

$$\frac{A_t}{B_t} = \frac{K_t}{mK_0(1+r)^t} = \frac{K_0[(1+r)^t - 1]}{rmK_0(1+r)^t},$$

or

$$\frac{A_t}{B_t} = \frac{1-(1+r)^{-t}}{rm}. \tag{2}$$

In defining the ratio of replacement to depreciation $\frac{R_t}{A_t}$, we are interested in the magnitude of this ratio only for the values of $t \geqslant m$ because in periods $t < m$ there is no need for replacement since the lifespan m of capital investment objects has not yet been reached. In year $t = m$, it is necessary to replace the machines installed m years ago, and therefore the replacement requirements are K_0. In period $t = m+1$, the replacement requirements are $K_0(1+r)$ i.e., they equal the gross investments at the initial period, etc.

Generally, in period t ($t \geqslant m$), replacement requirements equal gross investments in period $t-m$, i.e. $K_0(1+r)^{t-m}$.

Therefore

$$\frac{R_t}{A_t} = \frac{K_0(1+r)^{t-m}}{\frac{K_t}{m}} = \frac{K_0(1+r)^{t-m}mr}{K_0[(1+r)^{t-1}]}.$$

Finally, after simple transformations, we have

$$\frac{R_t}{A_t} = \frac{rm}{(1+r)^m-(1+r)^{m-t}}. \tag{3}$$

This formula determines the ratio of the amount of replacement requirements to the amount of linear depreciation in period $t(t \geqslant m)$. In period $t = m$ the ratio of replacement to depreciation equals

$$\frac{R_m}{A_m} = \frac{rm}{(1+r)^m-1}. \tag{3a}$$

If the process of growth of gross investments is a continuous one (i.e. the periods in which additional investments are added to the existing stock of means of production tend to zero and, at the same time, the number of such periods tends to infinity), then[6]

$$(1+r)^m \to e^{rm},$$

and formula (3a) assumes the following form:

$$\frac{R_m}{A_m} = \frac{rm}{e^{rm}-1}. \tag{4}$$

---

[6] Let us assume that each of the m periods has been divided into n parts. Then, the total number of periods is mn and the rate of growth of additional investments corresponding to these new periods is r/n. Therefore

$$(1+r)^m = \left(1+\frac{r}{n}\right)^{mn} = \left[\left(1+\frac{r}{n}\right)^n\right]^m$$

$$= \left[1+n\cdot\frac{r}{n}+\frac{n(n-1)}{2!}\cdot\frac{r^2}{n^2}+\frac{n(n-1)(n-2)}{3!}\cdot\frac{r^3}{n^3}+\dots\right]^m$$

$$= \left[1+r+\frac{n-1}{2n}r^2+\frac{(n-1)(n-2)}{6n^2}r^3+\dots\right]^m$$

$$= \left[1+r+\frac{1-\frac{1}{n}}{2}r^2+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{6}r^3+\dots\right]^m \to$$

$$\to \left[1+r+\frac{r^2}{1\cdot 2}+\frac{r^3}{1\cdot 2\cdot 3}+\dots\right]^m = e^{rm} \text{ (when } n\to\infty).$$

It can be seen from formula (4) that the ratio $\frac{R_m}{A_m}$ depends then upon the rate of growth of investment r and the m lifespan of capital objects, or, more exactly, upon the product of these coefficients, i.e. rm.

Let us note that $\frac{R_m}{A_m} = 1$ only when $r = 0$. It is found that if we substitute $r = 0$ in formula (4), then $\frac{R_m}{A_m} = \frac{0}{0}$, i.e. the ratio becomes an indeterminate expression. However, the indeterminateness of this expression can be eliminated by applying de l'Hospital's formula according to which

$$\lim_{r \to 0} \frac{R_m}{A_m} = \lim_{r \to 0} \frac{R'_m}{A'_m} = \lim_{r \to 0} \frac{m}{me^{rm}} = \frac{m}{m} = 1 .$$

This result confirms the fact established by us earlier that, in the case of simple reproduction ($r = 0$), the amount of replacement requirements equals depreciation. In other cases, though, ($r \neq 0$), $A_m \neq R_m$.

If the rate of growth in investment $r > 0$ (i.e. gross investments increase), then[7] $\frac{R_m}{A_m} < 1$, which means that replacement requirements are less than depreciation.

If however, $r < 0$ (i.e. gross investments decrease), then $\frac{R_m}{A_m} < 1$ and, thus, depreciation does not meet replacement requirements.

In his work on the theory of economic growth, Domar[8] produces a table showing the magnitude of the ratios $\frac{A_m}{B_m}, \frac{R_m}{B_m}, \frac{R_m}{A_m}$[9]

[7] We found this to be so from the numerical example considered at the beginning of this chapter.

[8] E. D. Domar, *Essays in the Theory of Economic Growth*, New York, Oxford University Press, 1957.

[9] It is easy to show that the ratio $y = \frac{R_m}{A_m}$ is a decreasing function of variable $x = rm > 0$. Indeed, if $y = \frac{x}{e^x - 1}$ then $y' = \frac{e^x(1-x)-1}{(e^x-1)^2}$. It can be seen that $y' < 0$ both for $0 < x \leqslant 1$ and for $x > 1$.

depending upon the product rm; these ratios have been calculated on the basis of formulae corresponding to a continuous process. The Domar table is shown below:

rm	$\frac{A_m}{B_m}$	$\frac{R_m}{B_m}$	$\frac{R_m}{A_m}$	rm	$\frac{A_m}{B_m}$	$\frac{R_m}{B_m}$	$\frac{R_m}{A_m}$
	in percentages				in percentages		
0.1	95	91	95	1.5	52	22	43
0.2	91	82	90	2.0	43	14	31
0.3	86	74	86	2.5	37	8	22
0.4	82	67	81	3.0	32	5	16
0.5	79	61	77	3.5	28	3	11
1.0	63	37	58				

It follows from this table that if, for instance, $rm = 0.1$ then $\frac{R_m}{A_m} = 95$ per cent, or $R_m = 0.95\ A_m$. This means that 95 per cent of the whole depreciation reserve should be used up for replacement and 5 per cent remains for net investment.

These ratios are completely different when, for instance, $rm = 3.5$, for, then, only 11 per cent of the depreciation reserve is used up for replacement and 89 per cent remains for net investment.

Domar gives some figures obtained from statistical records concerning the actual situation in the United States. He surmises that in the United States the rate of growth of additional gross investment r is about 3 per cent and the average lifespan m of capital investment objects is about 30 years. Therefore, according to Domar, for the USA the product rm is $0.03 \times 30 = 0.9$, or, roughly, $rm = 1$. For this value of the product rm only 58 per cent of the depreciation reserve is used for replacement and 42 per cent constitutes a reserve which can be allocated to net investment.

On the basis of data for the national economy of the Soviet Union, Domar assumes that in that country in the period 1930–1950, the average lifespan of capital investment objects was, as in the United States, about 30 years,[10] and the

[10] It can be surmised, on this basis, that the levels of production technique in the Soviet Union and in the United States were fairly close. It could also be assumed that even if the level of technique in the Soviet Union was, at

rate of growth of investment was much higher than in the United States since it amounted to 0.12 or 12 per cent.

Thus, it may be assumed that the product rm for the Soviet Union in the years 1930–1950 was $0.12 \cdot 30 = 3.6$, or after rounding off, 3.5. This would mean that only 11 per cent of the depreciation reserve was allocated to replacement and the remainder, i.e. 89 per cent was used for investment. It follows that in the Soviet Union depreciation reserves were an important source for the financing of investments and the development of the national economy.[11]

It may be that Domar's estimate of the product rm, and particularly of the rate of growth of investment is for the Soviet Union too high. If a more realistic assumption is accepted, namely, that $r = 0.08$ and, thus, $rm = 2.5$, then, in any case, only 22 per cent of the depreciation reserve is used up for replacement and the remaining 78 per cent is earmarked for investment.

Let us now try to derive a formula for determining the amount of depreciation equalling exactly replacement requirements. Then, even in the case of expanded reproduction $\frac{R_m}{A_m} = 1$, i.e. the amount written off equals the replacement requirements.

It is found that the rate of depreciation a that satisfies this condition is defined by the following formula:

$$a = \frac{r}{e^{rm}-1}. \tag{5}$$

Indeed, the rate of depreciation a equals the ratio of depreciation A_m to the value of the capital object K_m, i.e. $a = \frac{A_m}{K_m}$. If we postulate that the equality $A_m = R_m$ must hold and considering that:

that time, lower than in the United States, nevertheless technical equipment in the Soviet Union is newer and its lifespan in both countries is approximately the same.

[11] This source of investment is somehow concealed since it is not shown in economic statistics. As a rule, only a surplus of gross accumulation over depreciation is taken as net accumulation. Strumilin pointed out that depreciation reserves in industry are also a source for the financing of the economic development of the Soviet Union. See S. S. Strumilin, *La Planification en USSR*, Paris, 1947, p. 31.

(1) the size of the replacement fund at moment m must equal investments $B_0 = K_0$ at the initial period, and, therefore, $R_m = K_0$,

(2) $$K_m = \frac{e^{rm}-1}{r} \cdot K_0,$$ [12]

we obtain

$$a = \frac{A_m}{K_m} = \frac{R_m}{K_m} = \frac{K_0}{K_0 \frac{e^{rm}-1}{r}} = \frac{r}{e^{rm}-1}.$$

It is found that for the rate of depreciation $a = \frac{r}{e^{rm}-1}$, depreciation equals replacement requirements. The rate of depreciation defined by formula (5) is called *the actuarial rate of depreciation*. Let us note that the reciprocal of the actuarial rate of depreciation, i.e. the expression $\frac{e^{rm}-1}{r}$, constitutes the end value of *a guaranteed unit annuity* after m years at the compound rate of interest r.[13]

If, then, the annual payment from a guaranteed annuity is $\frac{r}{e^{rm}-1}$, its end value after m years at the compound rate of interest r is exactly 1.

Let us now consider the economic meaning of the actuarial rate of depreciation. We know that in the case of expanded reproduction, the value of a capital object increases, and if the rate of

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[12] The value of the capital investment object in period m in a step process is defined by formula (1) or (1a). Switching to a continuous process, as was done to determine the ratio $\frac{R_m}{A_m}$, we obtain: $K_m = \frac{e^{rm}-1}{r} K_0$.

[13] *A guaranteed unit annuity* is a sequence of monetary units payable every year. The end value of such an annuity after m years at the compound rate of interest r is:

$$s_m = 1+(1+r)+(1+r)^2+ \dots +(1+r)^{m-1} = \frac{(1+r)^m-1}{r}.$$

This formula for a continuous process (i.e. when the length of the unit time tends to zero) assumes the form:

$$s_m = \frac{e^{rm}-1}{r}.$$

growth of additional investments is r, the increase in the value of the capital object[14] corresponds to the end value of a unit annuity at the rate of interest r (see formula (1)). If the depreciation reserve is formed on the basis of the actuarial rate, this reserve increases in the same way as the value of the capital object.

The problem of determining the amount of depreciation to be written off may become even more complicated. So far, we have assumed that each capital object remains for instance in productive use exactly m years, and after this period its use value drops to zero at once. This assumption can be changed by assuming that the value of the capital object and its productive efficiency gradually decline, or, what amounts to the same thing, that each year certain outlays must be made in order to maintain it in an unchanged state.

If we assume that part $k (0 \leqslant k \leqslant 1)$ of the value of a given capital object lasts to the end, and each year part $\frac{1-k}{m}$ of its value "disappears", then the ratio of replacement requirements to depreciation can be determined by the following formula:

$$\frac{R_m}{A_m} = \frac{rm}{e^{rm}-1} \cdot k-(1-k). \tag{6}$$

The expression $\frac{rm}{e^{rm}-1} k$ has been obtained by using the previous method of calculation (formula (4)) for the k-th part of the value of the object, i.e. the part which by assumption lasts to the very end—over the period of m years.

The expression $1-k$ denotes an annual decrease in the value of the capital object calculated on the assumption of a uniform wear and tear in time. Indeed, if within one year part $\frac{1-k}{m}$ of the value of a capital object is worn out, then within m years, the worn out part of the value of the capital object is $\frac{1-k}{m} m = 1-k$.

[14] And also the end value of the capital object because, by assumption, in the starting period there is no capital and the starting point is the investment $B_0 = K_0$, $B_1 = K_0(1+r)$, $B_2 = K_0(1+r)^2$ ...

Another and more serious complication involved in determining the amount of depreciation to be written off arises when we reject the unrealistic assumption that all capital objects of a given enterprise, or of the whole economy, have the same lifespan. It is obvious that buildings for instance can be used longer than machines or some equipment which, in turn, have different lifespans.

The question arises in what way differences in the durability of various types of capital objects affect the problem of replacement and depreciation. This problem is dealt with by a special branch of mathematics called the *theory of replacement*. The historical starting point for these considerations is the mathematical theory of changes in population, or mathematical demography. The theory of mathematical demography can be generalized and applied to other populations to which (as in a human population) certain elements are added and in which other elements disappear, thus causing changes in its size and structure.

It has been established that the generalized theory of changes in human population can easily and usefully be applied to animal populations. By making certain assumptions concerning, for instance, the vegetation conditions and fertility of fishes in a pond, it is possible to determine changes in the fish population of the pond depending upon the time and extent of fishing. On this basis, optimal fishing norms can be determined and this, of course, is of great practical importance to fish breeding.

The generalized theory of mathematical demography can, of course, be successfully applied not only to a population of live animals but also to a collection of objects which in the course of time become worn out and must be replaced. A set of means of production may be treated as a population in which there is a "death rate" and a "birth rate" of its elements. "Dying" elements are the ones that are worn out during the process of production and must be removed from it; "born" elements are the ones that are newly brought into the process of production.

On the basis of these premises, the principles of mathematical demography are now used extensively in *the theory of replacement*. In discussing this theory we shall use the language of demography. It should be remembered, however, that the principles outlined here are applicable to all populations whose elements "die out" or are "replaced".

Let us consider a population whose size is constant and equals N and whose structure is also constant. This will be the case of the simple reproduction of population.

Let us denote by $B(t)$ *the birth rate* in year t. It is the ratio of the number of persons born in that year to the total population. Thus, the number of persons born in year t is $NB(t)$.

Let us assume that we have a table representing the distribution of the population according to age.[15] We can determine from this table *the probability of survival to a specific age* τ which we shall denote by symbol $p(\tau)$. We calculate it as a ratio of the number of persons at age τ, which we denote by $N(\tau)$, to the number of persons at age 0:[16]

$$p(\tau) = \frac{N(\tau)}{N(0)}.$$

Thus, the number of persons born in year, t, who will survive to the age τ, is $NB(t)p(\tau)$.

We shall treat the probability of survival to age τ, i.e. $p(\tau)$, as a continuous function of τ, since the age of persons τ changes in a continuous way. Function $p(\tau)$ is a decreasing one, the probability of survival to a later age being smaller than the probability of survival to an earlier age. A schematic graph of function $p(\tau)$ is presented in Fig. 12.

The derivative of this function, $p'(\tau)$, determines the velocity with which the probability of survival to age τ decreases and, since the function $p(\tau)$ is a decreasing one, $p'(\tau) < 0$.

Let us now denote by $m(\tau)$ the probability that a person who survived to age τ will not survive to age $\tau+1$. This is *the intensity of the death rate* at age τ. We calculate it as the ratio of the number of persons who die at age τ to the number of persons who have survived to age τ.

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[15] Tables of this kind can also be compiled for animal populations, machine populations, building populations, etc.

[16] The probability of survival to a certain age, $p(\tau)$, can also be determined for a population whose size increases or decreases. A description of this kind of statistical methods can be found in any textbook on demographic statistics.

It is easy to establish that

$$m(\tau) = -\frac{p'(\tau)}{p(\tau)}.$$

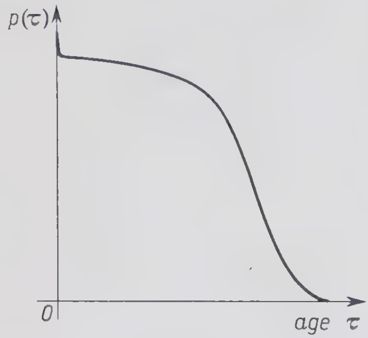

FIG. 12.

Indeed, if the number of persons in the population is N, then the number of persons who survived to age τ equals $Np(\tau)$. On the other hand, the decline in the number of persons at age τ in period dt is: $-Np'(\tau)dt$.[17] Thus, the decrease in the number of persons within one year ($dt = 1$) is $-Np'(\tau)$. In consequence

$$m(\tau) = -\frac{Np'(\tau)}{Np(\tau)} = -\frac{p'(\tau)}{p(\tau)}.$$

Since the number of persons born in year t is $NB(t)$, the number of persons born in year t and deceased at age τ is $-NB(t)p'(\tau) = NB(t)p(\tau)m(\tau)$.

The ratio of this number to the total number of persons born in year t is

$$\frac{NB(t)p(\tau)m(\tau)}{NB(t)} = p(\tau)m(\tau) = -p'(\tau).$$

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[17] Since $p(\tau) = \frac{N(\tau)}{N}$, we have $N'(\tau) = Np'(\tau)$ and therefore the decrease in the number of persons in period dt (i.e. the differential of function $N(\tau)$, is $-Np'(\tau)dt$. We put the minus sign in order to obtain a positive value of the number determining the decrease in the number of persons (since $p'(\tau) < (0)$).

This ratio does not depend upon t. It is called *the coefficient of elimination* or *the probability of death* at age τ and is denoted by $f(\tau)$.

The values of the coefficient $f(\tau)$ for various τ are given in elimination tables and they can be determined statistically directly from observations relating to the distribution of population by age and number of death at different ages. A schematic graph of function $f(\tau)$ is shown in Figure 13.

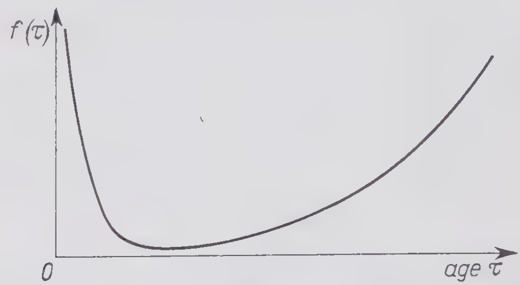

FIG. 13.

We shall now try to determine the number of persons deceased in any year t.

In year t, included in the population are persons born in the given year t and in previous years $t-1$, $t-2$, ...; if we denote by ω the upper human age limit (e.g. we assume that $\omega = 100$ years which in practice, with very few exceptions, is true), then the oldest persons in the population were born in year $t-\omega$.

The number of persons born in the years $t, t-1, t-2, \ldots, t-\omega$, is respectively: $NB(t)$, $NB(t-1)$, ..., $NB(t-\omega)$. Thus, the total number of persons deceased in period t, or *the elimination function* $V(t)$, will constitute the aggregate fall-off in the number of persons at different age groups:[18]

$$V(t) = NB(t)f(0) + NB(t-1)f(1) + \ldots + NB(t-\omega)f(\omega),$$

or

$$V(t) = N\sum_{t=0}^{\omega} B(t-\tau)f(\tau). \qquad (7)$$

[18] If t denotes, for instance, the current year, then $NB(t-3)f(3)$ denotes the number of persons who were born 3 years ago and died in the current year.

If, instead of a discrete process, we consider, continuous process of elimination of population and use the limit for defining $V(t)$ in place of a finite sum, we obtain the corresponding definite integral:

$$V(t) = N\int_0^\omega B(t-\tau)f(\tau)\,d\tau \tag{7a}$$

It follows from the formulae (7) and (7a) that the elimination function $V(t)$ shows how the amount of elimination in a given year (or at a given moment) t depends upon the year (or moment) in which the persons included in the population were born and upon the elimination coefficient in the period from t to $t-\omega$.

To offset the elimination in the population in year (moment) t, the number of persons born in this year should be the same as the number of persons deceased. Thus, $NB(t)$ must equal $V(t)$ and only then will the size of the population remain the same.

Therefore, the condition of simple reproduction, in which some elements leave the population and others join it, can be presented as follows:

$$NB(t) = N\int_0^\omega B(t-\tau)f(\tau)\,d\tau,$$

or after reducing by N:

$$B(t) = \int_0^\omega B(t-\tau)f(\tau)\,d\tau.^{19} \tag{8}$$

[19] Instead of ω (the maximum lifespan of the element of the population) we often use ∞. Then, the integral appearing on the right-hand side of equation (8) has the following form

$$\int_0^\infty B(t-\tau)f(\tau)\,d\tau.$$

This is a mathematical generalization of the above considerations and of equation (8), but it does not affect significantly the final conclusions. For, if there exists an upper limit for the length of life ω, we have

$$\int_0^\infty B(t-\tau)f(\tau)\,d\tau = \int_0^\omega B(t-\tau)f(\tau)\,d\tau + \int_\omega^\infty B(t-\tau)f(\tau)\,d\tau$$

$$= \int_0^\omega B(t-\tau)f(\tau)\,d\tau,$$

because the second integral in this sum equals zero.

This is the *replacement equation*; it is an equation of the integral type because the unknown function $B(t)$ determining the birth rate appears under the sign of integral. The integral equation (8) is solved by finding such a function $B(t)$ that, after being substituted in equation (8), it satisfies this equation (on the assumption that the elimination function $f(\tau)$ is known).

The method of solving integral equation (8) is similar to that used for solving differential equations. We assume that the solution of integral equation (8) has the form $B(t) = Qe^{\varrho t}$, where Q and ϱ are coefficients, and $Q \neq 0$.

Substituting $B(t) = Qe^{\varrho t}$ in equation (8) we obtain

$$Qe^{\varrho t} = \int_0^\omega Qe^{\varrho(t-\tau)} f(\tau)\, d\tau,$$

or

$$Qe^{\varrho t} = Qe^{\varrho t} \int_0^\omega e^{-\varrho t} f(\tau)\, d\tau.^{20}$$

After dividing both sides by $Qe^{\varrho t}$, we have the condition

$$\int_0^\omega e^{-\varrho t} f(\tau)\, d\tau = 1. \tag{9}$$

We can see that the function $B(t) = Qe^{\varrho t}$ may be a solution of equation (8) if equation (9) holds; the latter, as in solving differential equations, is called the characteristic equation of the integral equation (8).

We know from the theory of integral equations that there exists an infinite but denumerable number of values of ϱ which satisfy equation (9). Moreover, it can easily be established that if $\varrho_1, \varrho_2, \varrho_3 \ldots$ satisfy the characteristic equation (9) and, thus, functions $Q_1 e^{\varrho_1 t}$, $Q_2 e^{\varrho_2 t}$, $Q_3 e^{\varrho_3 t}$ ... are solutions of integral equation (8), then any linear combination of these functions is also a solution of this integral equation. Therefore, the general solution of integral equation (8) takes the form[21]

$$B(t) \sum_{i=1}^{\infty} Q_i\, e^{\varrho_i t}, \tag{10}$$

[20] Let us note that it is possible to remove factor $e^{\varrho t}$ before the integral because the integrated variable is τ and not t.

[21] This can be checked by substituting in equation (8) the general solution defined by formula (10).

We shall now solve the characteristic equation (9). The simplest method of solving it is the graphical one.

A real solution of equation (9) can easily be obtained, and—as can easily be ascertained—there is only one such solution.

To solve this equation we treat the integral appearing on the left-hand side of equation (9) as a function of parameter ϱ:

$$R(\varrho) = \int_0^\omega e^{-\varrho\tau} f(\tau)\, d\tau. \tag{11}$$

The integrated function is a product of two positive functions of which one, $f(\tau) \geqslant 0$, is independent of ϱ and the other, $e^{-\varrho\tau} > 0$ is a monotonically decreasing function of parameter ϱ. It follows that function (11) is a decreasing one.

It can easily be established, too, that when $\varrho \to +\infty$, $e^{-\varrho\tau} \to 0$ and, therefore, also $R(\varrho) \to 0$, and when $\varrho \to -\infty$, $e^{-\varrho\tau} \to \infty$, and, therefore, also $R(\varrho) \to +\infty$.

These conclusions enable us to draw a graph of function $R(\varrho)$, (see Figure 14).

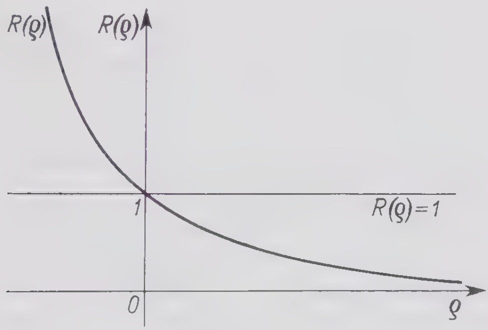

FIG. 14.

If we now draw the line $R(\varrho) = 1$, then according to equation (8), the abscissa of the point of intersection of the curve $R(\varrho) = \int_0^\omega e^{-\varrho\tau} f(\tau)\, d\tau$ with the line $R(\varrho) = 1$ will be a solution of equation (9). Because of the monotonic nature of function $R(\varrho)$, there is only one point of intersection and, therefore, only one real solution.

It can be shown that the point of intersection of the graph of function (11) and function $R(\varrho) = 1$ is situated on the axis of ordinates and, therefore, the real solution of equation (9) is $\varrho = 0$.

Indeed, we know that $\int_0^\omega f(\tau)d\tau = 1$. This follows from the nature of the coefficient of elimination $f(\tau)$. For, it is a fraction of the number of persons belonging to the population studied and deceased at age τ. It is evident that if "addition" or "integration" covers all possible age groups that can be reached by the persons included in the population, the sum or integral obtained will equal 1.[22]

Let us assume that $\varrho = 0$. Then $e^{-\varrho\tau} = 1$ and

$$R(\varrho) = \int_0^\omega e^{-\varrho\tau} f(\tau)d\tau = \int_0^\omega f(\tau)d\tau = 1.$$

Thus, the curve $R(\varrho) = \int_0^\omega e^{-\varrho\tau} f(\tau)d\tau$ intersects the axis of ordinates at point (0,1).

We have proved that the characteristic equation (9) can have only one real solution: $\varrho = 0$. Hence, the solution of equation (8) is a constant function. Indeed, for $\varrho_1 = 0$, we have $B(t) = Q_1 e^{\varrho_1 t} = Q_1 = \text{const.}$

This means that in the case of simple reproduction in a population the birth rate $B(t)$ must be constant. This solution is intuitively obvious because, if the size of the population is to be unchanged, the number of births per year must correspond to the number of deaths per year.

We shall now deal with a case in which solutions of the characteristic equation (9) are complex. In this case, we write: $\varrho = \alpha + i\beta$ and the formula for the solution (10) of the replacement equation (8) will assume the following form

$$B(t) = Qe^{(\alpha+i\beta)t} = Qe^{\alpha t}e^{i\beta t} = Qe^{\alpha t}(\cos\beta t + i\sin\beta t).$$

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[22] This can be shown in yet another way. The coefficient $f(\tau)$ denotes the probability that a person in a given population will die at age τ. Therefore, the probability that a person will die between the ages of 0 and ω equals unity.

It follows from this form of solutions of integral equations that in the process of replacement there appear cycles called *autonomous cycles of the replacement process.*

Autonomous cycles of the replacement process are a result of a phenomenon known in demography as *an echo phenomenon.* If, for example, during war years the number of births declines, then after some time, when the war generation reaches the reproductive age, the number of births will continue to decline. The echo phenomenon may repeat itself many times at even successive intervals. The echo phenomenon is often regarded as an effect of heredity because the replacement equation can also be used in studies on heredity.

The replacement equation was formulated for the first time in 1910 by the Italian mathematician, Vito Volterra[23], and was called "the integral equation of hereditary effects"; it appears in biological populations in the form of certain characteristics of the population that manifest themselves from time to time. They are phenomena whose rate of growth $B(t)$ depends upon previous states at a certain period ω, corresponding to the heredity period of given characteristics.

Volterra's equation was introduced in 1913 into demography by the American statistician and demographer, A. J. Lotka, and in 1933 into studies on the problem of the replacement of means of production.[24] At that time began the process of generalization of the methods of mathematical demography and of their introduction into the theory of replacement.

Let us now take a closer look at complex roots of the char-

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[23] See Vito Volterra, *Leçons sur les équations intégrales et les équations intégro-différentielles*, Paris, 1913, p. 151 and ff. These lectures were given in 1910.

[24] See the paper by A. J. Lotka, Industrial Replacement, *Skandinavisk Aktuarietidskrift*, 1933. See also the study by the same author, Contribution to the Theory of Self-renewing Aggregates with Special Reference to Industrial Replacement, *Annals of Mathematical Statistics*, 1939, and On an Integral Equation in Population Analysis, *ibid*, 1939. M. Fréchet, *Les ensembles statistiques renouvellés et remplacement industriel*, Paris, 1949. A systematic exposition of the theory of replacement and of mathematical problems related to the replacement equation is given in the work by W. Saxer, *Versicherungsmathematik*, Part 2, Berlin, 1958, Chapter 4. See also J. A. Ville, *Leçons sur la démographie mathématique*, Paris (n.d.).

acteristic equation (9). Let us substitute in this equation the complex root $\varrho = \alpha + i\beta$. We obtain the following equation:

$$\int_0^\omega e^{-(\alpha+i\beta)\tau} f(\tau)\,d\tau = 1,$$

or

$$\int_0^\omega e^{-\alpha\tau}(\cos\beta\tau - i\sin\beta\tau) f(\tau)\,d\tau = 1. \tag{9a}$$

This last equation will be satisfied if the real component on the left-hand side of the equation equals one and the imaginary component equals zero:

$$\begin{cases} \int_0^\omega e^{-\alpha\tau}\cos\beta\tau f(\tau)\,d\tau = 1, \\ \int_0^\omega e^{-\alpha\tau}\sin\beta\tau f(\tau)\,d\tau = 0. \end{cases} \tag{12}$$

It follows from the first of the above equations that $\alpha < 0$. Indeed, let us compare the integral $\int_0^\omega e^{-\alpha\tau}\cos\beta\tau f(\tau)\,d\tau$ with the integral $\int_0^\omega e^{-\varrho\tau} f(\tau)\,d\tau = 1$ studied before (in which, as we have seen, $\varrho = 0$ and $e^{-\varrho\tau} = 1$). For one of the integrated factors appearing in the first integral, the inequality $\cos\beta\tau < 1$ holds for $\tau \neq 0$.[25] It can be seen, by comparison, that for $\alpha \geqslant 0$, this integral is less than unity. It can equal unity only when its other integrated factor $e^{-\alpha\tau} > 1$, from which it follows that $\alpha < 0$.

This is a very important conclusion indicating that cyclical fluctuations in the birth rate $B(t) = Qe^{\alpha t}(\cos\beta t + i\sin\beta t)$ have a decreasing amplitude, i.e. that they are damped cycles.

Experience confirms this conclusion. Statistical studies of demographic phenomena show that the phenomenon of echo gradually fades away.

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[25] $\beta \neq 0$ because otherwise the solution would not be complex.

On the basis of equation (12) we shall establish that the period T of the birth rate cycle is shorter than the longest lifespan of the persons included in the population, i.e. $T < \omega$.

Indeed, let us note that the integrated function of the integral

$$\int_0^{\omega} e^{-\alpha t} \sin \beta \tau f(\tau) d\tau,$$

is a periodic function with the period[26] $T = 2\pi/\beta$ and with an increasing amplitude because (since $\alpha < 0$) $e^{-\alpha t}$ increases as t increases. This function is presented graphically in Figure 15.

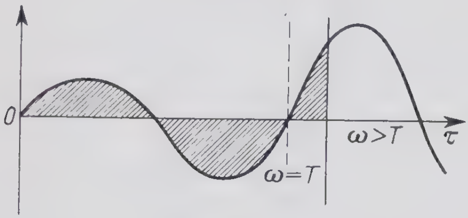

FIG. 15.

The value of the integral $\int_0^{\omega} e^{-\alpha \tau} \sin \beta \tau f(\tau) d\tau$ repiesents "the shaded area under the curve", in Fig. 15. It can easily be seen that in the interval [0, T] the shaded area below the axis of abscissae (i.e. the area to which a negative value is ascribed) is greater than the area above the axis of abscissae. This is so because the integrated function has an increasing amplitude. If, then, the length of the integration interval were $\omega = T$, the value of $\int_0^{\omega} e^{-\alpha \tau} \sin \beta \tau f(\tau) d\tau$ would be less than zero. It follows from condition (12) that this integral equals zero only when the length of the integration interval $\omega > T$, since only then can the area above the axis of abscissae (i.e. the positive area) be equal to the area below the axis of abscissae (i.e. the negative area).

[26] The period of a sinusoidal function is 2π and, therefore, in this case $T\beta = 2\pi$, hence $T = \frac{2\pi}{\beta}$.

We have thus shown that the fluctuations in the birth rate, corresponding to a complex solution of the characteristic equation (9), are of a decreasing nature with the fluctuation period $T < \omega$.

In conclusion, some comments of a general nature can be derived. Cycles of simple reproduction within a branch, as discussed above, i.e. *reinvestment cycles*,[27] may appear also in a socialist economy, in contrast to business cycles, caused by changes in profitability in various branches of production, unknown in a socialist economy.

A re-investment cycle may appear in our economy as a phenomenon of echo. During the period of the 6-year plan many new production establishments had been built. If we want to replace these establishments we may encounter a cumulation of reinvestments. Thus the heredity of the 6-year plan would extend into the future.

A similar situation prevails also in a capitalist economy with this difference that in the latter reinvestment cycles are superimposed on business cycles. Business cycles result in a cumulation of investments in certain years, and this, in turn, enhances business cycle fluctuations. This was duly noted by Marx.[28]

The most important conclusion to be drawn from our theoretical considerations is that replacement cycles, if they are not superimposed on business cycles, are of a decreasing nature. If there appears in a socialist economy a phenomenon of echo, which arises in consequence of an initial cumulation of investments, it gradually fades away and is eliminated from the development process of the national economy.

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[27] On the subject of re-investment cycles see J. Einarsen, *Reinvestment Cycles*, Oslo, 1938, and Tadeusz Czechowski, Cykliczność procesu reprodukcji prostej (Cycles in the Process of Simple Reproduction), *Scientific Series*, SGPiS, Warsaw, 1957.

[28] See K. Marx, *Capital*, vol. 2.

APPENDIX 1

DESCRIPTION OF SPECIAL MODELS DESIGNED TO SOLVE SOME PROBLEMS OF INTER-BRANCH FLOWS

IT is possible to construct models for solving automatically certain problems of inter-branch flows; their solution by ordinary computational methods could be quite cumbersome, especially when the number of branches of the national economy is large. Outlined below is a description of hydraulic and electrical models which can be used for solving the simplest problems of this type. It is possible, however, to construct similar models for solving more complex problems consisting, for instance, in obtaining optimal solutions in planning the allocation of accumulation, investments, etc.

Model 1 is used for determining the amounts of final products in particular branches, $x_1, x_2, \ldots, x_n$, when the aggregate amounts of products, $X_1, X_2, \ldots, X_n$ are given, or vice versa, for calculating $X_1, X_2, \ldots, X_n$ when $x_1, x_2, \ldots, x_n$ are given. A diagram of a hydraulic variety of such a model assuming that the national economy is divided into two divisions ($n = 2$), is shown in Figure 16.

The model has two upper starting containers; the amount of liquid in these containers is a measure of X_1 and X_2, i.e. of the aggregate products of Divisions 1 and 2, respectively. There are also four lower containers, denoted in the diagram by *A*, *B*, *C* and *D*. The upper and lower containers are interconnected by a system of pipes whose flow capacity (the bore of the pipe) may be regulated at will by an appropriate calibration of faucets. The model has no piston pumps for forcing the liquid into the pipes. The liquid flows from the upper to the lower containers only by gravity. The containers and the pipes may be made of glass to facilitate observation and the liquid may be coloured.

The functioning of the model is based on the following initial assumptions.

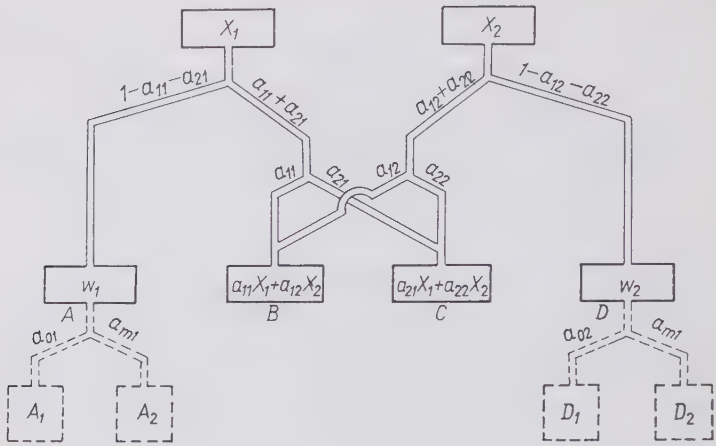

FIG. 16.

An ordinary table of inter-branch flows for two divisions (expressed in money-value units), as shown in Table 1, can be transformed into Table 2 by introducing inputs coefficients.

Table 1

x_{11}	x_{12}	x_1	X_1
x_{21}	x_{22}	x_2	X_2
w_1	w_2		
X_1	X_2		

Table 2

$a_{11}X_1$	$a_{12}X_2$	x_1	X_1
$a_{21}X_1$	$a_{22}X_2$	x_2	X_2
w_1	w_2		
X_1	X_2		

In these tables w_1 and w_2 denote the values added in Division 1 and Division 2, respectively. From Table 2 we derive the following initial equations for our further considerations:

$$\begin{cases} X_1 = a_{11}X_1 + a_{12}X_2 + x_1, \\ X_2 = a_{21}X_1 + a_{22}X_2 + x_2, \end{cases} \tag{1}$$

$$\begin{cases} X_1 = a_{11}X_1 + a_{21}X_1 + w_1, \\ X_2 = a_{12}X_2 + a_{22}X_2 + w_2. \end{cases} \tag{2}$$

The first two of these equations, called balance-sheets of production equations, have been obtained by adding up the elements in the rows of Table 2, and the last two equations, called cost outlay equations, by adding up the elements in the columns of Table 2.

If the inner diameters of the pipes of the model (or their flow capacity regulated by faucets) are proportional to the numbers that are written down beside the pipes in Fig. 15, then the liquid flowing out of containers 1 and 2 will be distributed, at each branch-off point, in proportion to the bore of the pipes into which the feeding pipe branches off. Thus, for example, the liquid from container 1 will be distributed at the first branch-off point into two flows at the ratio corresponding to the bores of the pipes, i.e. in the ratio: $(1-a_{11}-a_{21}):(a_{11}+a_{21})$.

It follows that the amount of liquid which will accumulate in the bottom container A will correspond to w_1, i.e. the value added in Division 1.[1]

The liquid flowing out of container 2 will be distributed in a similar way. At each consecutive branch-off point the liquid will be distributed according to the same principle, i.e. always in proportion to the bore of the pipes.

It is easy to ascertain that, as a result, the amount of liquid collected in the bottom container will correspond to $a_{11}X_1 + a_{12}X_2 = X_1 - x_1$, as follows from the first equation of the system of initial equations (1). Similarly, the amount of liquid collected in the bottom container C will correspond to $a_{21}X_1 + a_{22}X_2 = X_2 - x_2$.

It can be seen from the above considerations that if the amount of liquid poured into the upper containers is given, then, after completing the experiment and determining the amount of liquid in the bottom containers B and C, we can determine the amounts of the final products x_1 and x_2 by determining the differences in the amounts of liquid in the corresponding upper and lower containers $X_1 - (X_1 - x_1) = x_1$ and $X_2 - (X_2 - x_2) = x_2$. Thus, the first problem has been solved.

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[1] Indeed, after multiplying X_1 by $1-a_{11}-a_{21}$, i.e. by the fraction determining the amount of liquid that flows out of container 1 and accumulates in container A we obtain $X_1(1-a_{11}-a_{21}) = X_1 - a_{11}X_1 - a_{21}X_1 = w_1$, which agrees with the first equation of the system of initial equations (2).

The essence of the construction of an electrical model is identical with the hydraulic one described above. The role of the liquid in the electrical variety is played by an electric current of intensity proportional to X_1 or X_2, respectively. The pipes will thus be replaced by conductors with resistance R, so chosen that they are inversely proportional to the quantities determined in the hydraulic model by the bore of the pipes. This means that the conductivity $\frac{1}{R}$ of the corresponding connections should be proportional to the quantities determining the distribution of current intensity in the branches. Since the voltage of the electric network with which the model is connected is constant, we can measure the power of the flow of current, instead of its intensity, i.e. we can measure in watts instead of amperes.

On the basis of the measurement of intensity or power of the electric current, obtained by installing meters at the points of "inlet" or "outlet" of the model, we can, as with the use of a hydraulic model, determine the values of x_1 and x_2 when X_1 and X_2 are given.

More interesting is the application of the model, described above, for solving the opposite problem consisting in determining the total products X_1 and X_2 when the final products x_1 and x_2 are given. When a hydraulic model is used, before we open the upper containers, we pour into the lower containers B and C (Fig. 16) the amounts of liquid corresponding to the given quantities x_1 and x_2, and into the upper containers any amounts of liquid. Then, we open simultaneously the outlet pipes of the upper containers and allow the liquid to flow out until the amount of water in the container B (or C) equals the amount of water which has flowed out of the corresponding upper container. In order to time properly the moment at which the flow of the liquid must be stopped, the scale determining the level of the liquid in the upper container should be read off "from the top", i.e. from the initial level of the liquid, and the scale of the bottom container should be read off in the usual way, i.e. from the bottom of the container.

It is obvious that at the moment of turning off the flow from the upper containers the amount of liquid in the bottom containers B and C will equal X_1 and X_2, respectively. Indeed, "at the moment of equilibrium" the amount of liquid that has flowed out of the upper container X_1 equals the amount of liquid in container B, which means that the first equation of system (1)

$$X_1 = a_{11}X_1 + a_{12}X_2 + x_1$$

is satisfied. Similarly, the second equation of system (1) is satisfied.

As a by-product of this experiment we obtain a certain amount of liquid in the containers A and D which allows us to determine the value added w_1 and w_2 in Divisions 1 and 2.

In conducting the experiment designed to determine X_1 and X_2 when x_1 and x_2 are given by means of a hydraulic model, certain practical difficulties of a technical nature arise. For example, for the experiment to be successful it is necessary that the speed of flow of the liquid through pipes of different diameter should be the same.

In an electrical model this difficulty does not arise because the time of flow of the current is negligible. In this kind of model it is also easier to solve the problem of catching "the moment of equilibrium" because the current can be cut-off at that moment automatically. It would be necessary, however, to install at the "exit" points of the model additional sources of current that would supply the exit points B and C with the current of an intensity (or power) corresponding to x_1 and x_2, respectively. The moment of equilibrium is reached when the intensity (or power) of current on the exit meter at point B (or C) equals the drop in the intensity (or power) of current on the entry meter at point X_1 (or X_2).

It is not difficult to imagine how these models can be developed to enable us to determine final products when total products are given (or vice versa) when the national economy is divided into more than two branches. A hydraulic model has rather technically limited possibilities of development because it would require an intricate system of many pipes of various sizes and numerous ramifications arranged in such a manner that the velocity of flow of the liquid is more or less the same; this would be very difficult to accomplish.

With an electrical model these difficulties do not arise and it is a relatively simple task to construct a model for solving the problem given above when the economy is divided into n branches.

Let us note that model 1 can be further developed so as to enable us to divide the value added w_1 and w_2 into wages and the the surplus according to the equations:

$$\begin{cases} w_1 = a_{01}X_1 + a_{m1}X_1, \\ w_2 = a_{02}X_2 + a_{m2}X_2, \end{cases} \tag{3}$$

where a_{01} and a_{02} are wage coefficients and a_{m1} and a_{m2} are profit coefficients of Division 1 and Division 2, respectively. To achieve this, two pipes with inner diameters proportional to a_{01} and a_{m1} (or a_{02} and a_{m2}) should be installed to connect container A (or D) with additional containers A_1 and A_2 (or D_1 and D_2) (Fig. 16). By admitting the liquid into these additional containers, we obtain the sum of wages and profits in both divisions of the national economy.

To conclude the description of model 1, let us note that it can be built only on the assumption that all the coefficients appearing in initial equations (1) and (2) or in the additional ones (3), are positive or equal zero. In the latter case, the bore of the corresponding pipe in a hydraulic model or the conductivity of the corresponding conductor in an electric model would have to equal zero. Without some additional equipment, this model cannot be used in cases for which some coefficients in the initial equations are negative. One can jokingly say that the model is not suitable "for planning deficits" (the case of $a_m < 0$) in any branch of the national economy.

Model 2 is designed to solve certain problems connected with the process of economic growth which we have presented in the form of differential equations (Chapter 4).

Let us compile a table of reproduction and investment flows and write the initial equations for the studied growth process, on the assumption that the national economy consists of two branches of production.

x_{11}	x_{12}	I_{11}	I_{12}	$x_1^{(0)}$	X_1
x_{21}	x_{22}	I_{21}	I_{22}	$x_2^{(0)}$	X_2
w_1	w_2				
X_1	X_2				

After introducing technical coefficients a, investment coefficients b and the rate of consumption k, this table will assume the following form:

c	$a_{11}X_1$	$a_{12}X_2$	$b_{11}\frac{dX_1}{dt}$	$b_{12}\frac{dX_2}{dt}$	$x_1^{(0)}$	X_1
	$a_{21}X_1$	$a_{22}X_2$	$b_{21}\frac{dX_1}{dt}$	$b_{22}\frac{dX_2}{dt}$	$x_2^{(0)}$	X_2
v	$a_{01}X_1$	$a_{02}X_2$				
m	$b_{11}\frac{dX_1}{dt}$	$b_{12}\frac{dX_2}{dt}$				
	$b_{21}\frac{dX_1}{dt}$	$b_{22}\frac{dX_2}{dt}$				
	$k_{m1}X_1$	$k_{m2}X_2$				
	X_1	X_2				

In this table k_{m1} and k_{m2} denote the parts of the value of the total products X_1 and X_2 that constitute profit earmarked for consumption.

The first pair of the initial equations of the growth process under consideration can be written in the form of differential equations:[2]

$$\begin{cases} X_1 = a_{11}X_1 + a_{12}X_2 + b_{11}\dfrac{dX_1}{dt} + b_{12}\dfrac{dX_2}{dt} + x_1^{(0)}, \\ X_2 = a_{21}X_1 + a_{22}X_2 + b_{21}\dfrac{dX_1}{dt} + b_{22}\dfrac{dX_2}{dt} + x_2^{(0)}. \end{cases} \tag{4}$$

[2] Let us note that instead of the derivatives $\frac{dX_1}{dt}$ and $\frac{dX_2}{dt}$ appearing in tables of investment flows and in equations (4), we can introduce finite differences ΔX_1 and ΔX_2 corresponding to the increments of the total products in a given period and we can treat such a process as a discrete one. This does not affect further reasoning.

Denoting, in turn, the rates of growth of the total product in Divisions 1 and 2 by r_1 and r_2, respectively,

$$r_1 = \frac{\frac{dX_1}{dt}}{X_1} \quad \text{and} \quad r_2 = \frac{\frac{dX_2}{dt}}{X_2},$$

we can transform the differential equations (4) into ordinary equations:

$$\begin{cases} X_1 - x_1^{(0)} = (a_{11}+b_{11}r_1)X_1 + (a_{12}+b_{12}r_2)X_2, \\ X_2 - x_2^{(0)} = (a_{21}+b_{21}r_1)X_1 + (a_{22}+b_{22}r_2)X_2, \end{cases} \tag{5}$$

which indicate that the part of the total product which is not consumed in a given division, i.e. gross investment, is partly used up for reproduction and partly for investments in Division 1 and Division 2.

In a similar way, adding up the first two columns of the table of reproduction and investment flows, we obtain the second pair of initial equations:

$$\begin{cases} X_1 = (a_{11}+a_{21})X_1 + (b_{11}+b_{21})r_1X_1 + a_{01}X_1 + k_{m1}X_1, \\ X_2 = (a_{12}+a_{22})X_2 + (b_{12}+b_{22})r_2X_2 + a_{02}X_2 + k_{m2}X_2. \end{cases} \tag{6}$$

The equations (5) and (6) are the starting point for constructing model 2 and for explaining its functioning. The diagram of the model is presented in Fig. 17. This model can be used for solving various problems; two such problems are described below.

In the first problem we assume that the following elements are given: the total products of Division 1 and Division 2, i.e. X_1 and X_2, the technical coefficient a, the investment coefficient b, the rate of consumption k and the rates of growth of total products in both divisions r_1 and r_2; we have to calculate the remaining elements and primarily the final products $x_1^{(0)}$ and $x_2^{(0)}$ which remain for consumption and for exports.

The functioning of model 2 is identical with the functioning of model 1, described above. In Fig. 17 in which the diagram of Model 2 is presented, the inner diameters (bores) of the pipes are shown. The liquid poured into the upper containers in amounts X_1 and X_2, respectively, flows through the system of pipes into the two bottom containers and accumulates in them in amounts

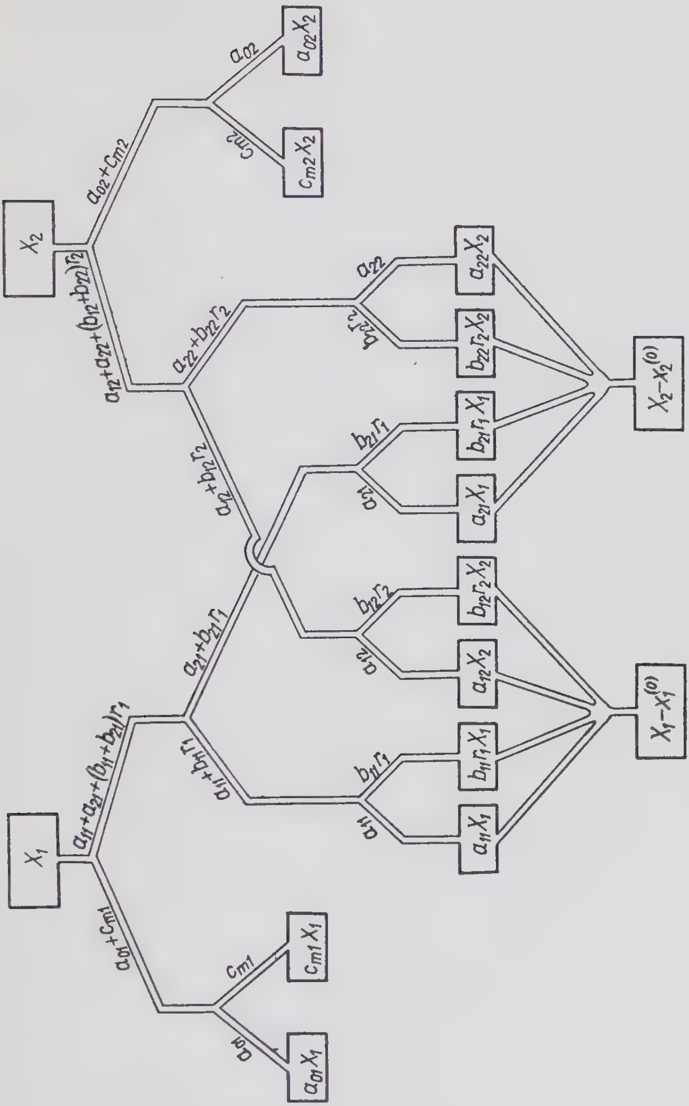
X_1
X_2
$a_{02}+c_{m2}$
a_{02}
c_{m2}
$a_{02}X_2$
$c_{m2}X_2$
$a_{12}+a_{22}+(b_{12}+b_{22})r_2$
$a_{22}+b_{22}r_2$
a_{22}
$b_{22}r_2$
$a_{22}X_2$
$b_{22}r_2X_2$
$a_{12}+b_{12}r_2$
$b_{21}r_1$
a_{21}
$b_{21}r_1X_1$
$a_{21}X_1$
$X_2-X_2^{(0)}$
$a_{21}+b_{21}r_1$
$b_{12}r_2$
a_{12}
$b_{12}r_2X_2$
$a_{12}X_2$
$a_{11}+a_{21}+(b_{11}+b_{21})r_1$
$a_{11}+b_{11}r_1$
$b_{11}r_1$
a_{11}
$b_{11}r_1X_1$
$a_{11}X_1$
$X_1-X_1^{(0)}$
$a_{01}+c_{m1}$
c_{m1}
a_{01}
$c_{m1}X_1$
$a_{01}X_1$

FIG. 17.

$X_1 - x_1^{(0)}$, and $X_2 - x_2^{(0)}$, as follows from the equations (5). The difference between the initial amount of liquid in the corresponding top and bottom containers will enable us to determine the quantities $x_1^{(0)}$ and $x_2^{(0)}$ sought.

The second problem is the reverse of the first one. We assume that, besides the given technical coefficient a, investment coefficient b, the rate of consumption k and the rate of growth r, we have determined in advance the planned level of consumption, i.e. the quantities of the products of Divisions 1 and 2 earmarked for consumption: $x_1^{(0)}$ and $x_2^{(0)}$. We must determine the total products of both divisions, X_1 and X_2, necessary for achieving the planned targets. This problem can be solved with the help of Model 2 in the same manner as when Model 1 was used. Before starting the experiment, we pour the liquid into the bottom containers in amounts corresponding to $x_1^{(0)}$ and $x_2^{(0)}$, respectively, and into the top containers any amount of liquid. Then, we turn on the taps of the top containers and let the liquid flow until its amount in one of the bottom containers (say, the one on the left-hand side in Fig. 17) is the same as the amount of liquid that has flowed out of the corresponding top container (also the one on the left-hand side). At this moment, we turn off the taps of the top containers. It can easily be seen that at this moment the equations (5) are satisfied and therefore the amounts of liquid which have flowed out of the top containers determine X_1 and X_2.

All the comments pertaining to the technical design of Model 1, its adaptation to make it suitable for solving problems arising in connection with a larger number of branches of the economy, as well as the comments on the construction of an electrical model 1 apply also to Model 2.

As we have already mentioned, Models 1 and 2 can be further developed so as to make them suitable for solving other problems relating to economic planning. It would be interesting, for example, to determine with the help of Model 2 the rates of growth of the national product, r_1 and r_2, when the total products X_1 and X_2 as well as $x_1^{(0)}$ and $x_2^{(0)}$ are given.

The problem can be solved with the help of Model 2 by the method of trial and error, by changing the bores of the pipes (or the resistance in the electric model) whose diameters are determined by the coefficients r_1 or r_2. The problem is solved when X_1 and X_2 of liquid, poured into the top containers, collect in the bottom containers (Fig. 17) in quantities of exactly $X_1 - x_1^{(0)}$ and $X_2 - x_2^{(0)}$, changing the diameters of the pipes until this condition is satisfied. If the magnitudes of other coefficients are known, then the diameters of the corresponding pipes thus adjusted will enable us to determine r_1 and r_2.

We can install in the containers, placed between the top and bottom containers (Fig. 17) special meters for measuring the quantity of liquid flowing through them. We can read off from them the amounts of reproduction and investment outlays needed for attaining the targets of total production X_1 and X_2, or of consumption $x_1^{(0)}$ and $x_2^{(0)}$. For instance, $a_{11}X_1$ are the reproduction outlays in Division 1 in the form of the products of this division; $a_{12}X_2$ are the reproduction outlays in Division 1 in the form of the products of Division 2. Similarly, $b_{11}r_1X_1$ are the investment outlays in Division 1 in the form of the products of this division, and $b_{12}r_2X_2$ are the investment outlays in Division 1 in the form of the products of Division 2. As a result, we can read off from the Model both the divisional and material structure of reproduction and investments. When the electric model is used, meters must be installed for measuring the intensity (or power) of the current instead of containers used in the hydraulic model.

The containers shown in Fig. 17 on both sides represent the wage funds in both divisions of the national economy, $a_{01}X_1$ and $a_{02}X_2$, and the parts of the value of the product of each division $k_{m1}X_1$ and $k_{m2}X_2$ constituting profit earmarked for consumption.

Models for solving certain economic problems were designed in the past, for instance, by Lerner and Phillips—to present the mechanism of the functioning of the Keynesian model.[3] The aim was, for instance, to determine the amount of investment *i* for an *a priori* given level of national income and a given propensity to consume. Lerner's model is much more complex in comparison with the models described above since it consists not only of pipes through which the liquid flows by gravity, but also of a system of pumps for circulating the liquid. Phillips' model is electrical.

[3] See A. W. Phillips, Mechanical Models in Economic Dynamics, *Economica*, 1950. Another model designed in Switzerland is described in *Jahrbücher für Nationalökonomie und Statistik*, 1958.

The following comments can be made in relation to the possibilities of constructing mechanical models for solving certain economic problems.

At present, especially now that electronic computers are applied to numerical calculations, the possibilities of solving numerically complex economic problems are infinitely greater. This fact is strictly related to the problem of centralization and decentralization of management. In recent years, in our economic practice, there is a trend toward decentralization in management. One of the arguments, but not the only one, for decentralization in management were the difficulties of solving a huge complex problem by one central authority. It was necessary to allocate the tasks of economic planning to local centres and individual enterprises.

It is likely, however, that the mechanization of computations and appropriate electronic computers will make possible greater centralization of management of the economy. Even today, electronic computers are used for solving systems of linear equations with a practically unlimited number of unknowns; with their help matrices of a system of linear equations are inverted.[4]

We can thus see that the technical revolution in electronic computers creates new conditions for the management of the economy and makes centralized management quite feasible. This does not mean, however, that the development of the methods of management must go in the direction of centralization, because there are other considerations, and particularly participation of workers in the management of a socialist economy, the problem of workers' councils and of the democratization of production relations, which make desirable a certain degree of decentralization in the management of a socialist economy.

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[4] It is worth mentioning that even Hayek and Robbins (see *Collectivist Economic Planning*, ed. F. A. Hayek, London, 1935, pp. 212 and ff) doubted if it would be possible to manage centrally the national economy because it would necessitate solving a tremendous number of equations which, at that time, was practically impossible.

APPENDIX 2

SOME NOTES ON DIFFERENTIAL AND DIFFERENCE EQUATIONS[1]

by

A. BANASIŃSKI

1. *Definition and classification of differential equations*

We define as *ordinary differential equations* those equations which express the relation between an independent variable x, an unknown function (a dependent variable) y and its derivatives of various orders $y', y'', \ldots, y^{(n)}$.

According to this definition we can write an ordinary differential equation of the nth order in the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{1}$$

or, using Leibnitz's notation instead of Lagrange's:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0.$$

There are various types of differential equations. In the first place, we distinguish between equations of the first, second ... nth order, depending on the highest order of the derivative of the function $y = f(x)$ appearing in equation (1). An ordinary differential equation is thus of the nth order if the highest order of the derivative of function $y = f(x)$ is n.

[1] This appendix is intended to make it easier for readers to understand the mathematical passages contained in this book without further reference to any special textbooks on higher mathematics. The inclusion of this appendix is all the more desirable since a course of higher mathematics as taught at higher schools of economics does not include the theory of differential and difference equations.

We may thus write an ordinary differential equation of the first order in the form

$$F(x, y, y') = 0, \tag{1a}$$

a differential equation of the second order in the form

$$F(x, y, y', y'') = 0 \text{ etc.} \tag{1b}$$

There are certain differential equations which are called "incomplete"; these are the equations in which e.g., an independent variable x, or a function y or some derivatives of orders lower than that of a given differential equation do not appear *explicit.* Below are examples of "incomplete" differential equations of the first order:

$$F(x, y') = 0, \quad F(y, y') = 0, \quad F(y') = 0.$$

It should be noted that in the second and third equations, although the independent variable does not appear distinctly, it is, none the less, *implicit* in the function y and its derivative y'.

Below are further examples of "incomplete" differential equations of the second order:

$$F(x, y'') = 0, \quad F(y, y'') = 0, \quad F(y', y'') = 0,$$
$$F(x, y, y'') = 0, \quad F(x, y', y'') = 0.$$

From among the various kinds of differential equations, *linear differential equations* of the nth order, frequently used in practice, should be especially noted. They can be presented in the form[2].

$$a_0(x)y^{(n)}+a_1(x)y^{(n-1)}+ \dots +a_{n-1}(x)y'+a_n(x)y = b(x), \tag{2}$$

where $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ and $b(x)$ are functions of the variable x or constants.

In particular the differential equation

$$a_0(x)y'+a_1(x)y = b(x)$$

is a linear differential equation of the first order.

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[2] It should be noted that in a linear differential equation of the nth order the function y and its derivatives y', y'', ... $y^{(n)}$ appear in the first degree. Hence the description "linear". If the function y or its derivatives appear in a differential equation in a power higher than the first power and the highest power, in which there appears y or any one of its derivatives, is equal to r, then we say that the differential equation is of the degree r. It follows that the order of a differential equation is to be carefully distinguished from its degree.

Dividing the differential equation $a_0(x)y'+a_1(x)y = b(x)$ by $a_0(x) \neq 0$ and denoting $a_1(x)\dfrac{a_1(x)}{a_0(x)}$ by $p(x)$ and $\dfrac{b(x)}{a_0(x)}$ by $q(x)$ we arrive at an equation of the form

$$y'+p(x)y = q(x). \tag{2a}$$

It is in this form that *differential equations of the first order* are most frequently written.

If, in a linear differential equation a "free term" (i.e., an expression which does not contain a function y or its derivatives) equals zero, i.e. $b(x) = 0$, then this equation is called a *homogeneous* equation. Hence the general form of a *differential homogeneous linear equation of the* nth *order* is as follows:

$$a(x)y^{(n)}+a_1(x)y^{(n-1)}+ \dots +a_{n-1}(x)y'+a_n(x)y = 0. \tag{3}$$

Therefore, homogeneous linear differential equations of the first order take the form:

$$y'+p(x)y = 0. \tag{3a}$$

If in equation (2) or (3) the functions $a_0(x), a_1(x), \dots, a_n(x)$ are constant numbers, then the equations (2) and (3) are called *linear differential equations* (homogeneous or nonhomogeneous) *with constant coefficients*.[3]

We shall be mainly concerned with simple types of ordinary differential equations, more frequently met with in practice, and with methods of solving them in later sections of this appendix.

In the theory of differential equations and in its practical applications we have to deal with sets of differential equations containing two or more unknown functions (dependent variables).

Let us assume that y and z are different functions of the same dependent variable x, i.e., $y = f(x)$ and $z = g(x)$. There can then be differential equations containing two unknown functions. The following is an example of a differential equation of the first order of this type:

$$F(x, y, z, y', z') = 0. \tag{4}$$

[3] We deal with this type of equations and sets of equations in Chapter IV.

As a rule, in order to find the functions $y=f(x)$ and $z=g(x)$ two equations of this type, i.e. *a set of differential equations* must be given:

$$\begin{cases} F_1(x, y, z, y', z') = 0. \\ F_2(x, y, z, y', z') = 0. \end{cases} \tag{5}$$

Similarly, we can arrive at systems of ordinary differential equations of any order containing any number of unknown functions.

A separate class of differential equations is formed by *differential equations with partial derivatives.* In these equations there are two or more independent variables $x_1, x_2, \ldots, x_n$, and one (or more) dependent variables $y_1, y_2, \ldots, y_n$, which are functions of the independent variables $x_1, x_2, \ldots, x_n$, as well as partial derivatives of these functions of several variables. We shall not deal here with the theory of differential equations with partial derivatives. We shall just give an example of differential equations with partial derivatives of the first order, containing two independent variables x_1 and x_2 and one function $y=f(x_1, x_2)$.

This equation can be written as follows:

$$F(x_1, x_2, y, y'_{x_1}, y'_{x_2}) = 0.$$

And here is an example of a differential equation with partial derivatives of the second order:

$$F(x_1, x_2, y, y'_{x_1}, y'_{x_2}, y''_{x_1x_1}, y''_{x_1x_2}, y''_{x_2x_2}) = 0.$$[4]

2. *Solving a differential equation*

The solution of an ordinary differential equation (1) consists in finding a function $y=f(x)$ which satisfies this equation.

For instance, the solution of the differential equation $y'' + y = 0$ (which, as can easily be seen, is an incomplete differential equation of the second order since it contains neither x nor y', i.e., it does not contain the independent variable x or the first derivative function y *explicit*) is the function $y = \sin x$ (or, as

[4] The symbols for partial derivatives used here may be replaced by other commonly used symbols, viz.,

$$y'_{x_1} = \frac{\partial y}{\partial x_1}, \; y'_{x_2} = \frac{\partial y}{\partial x_2}, \; y''_{x_1x_1} = \frac{\partial^2 y}{\partial x_1^2}, \; \ldots$$

we frequently say, the curve of the equation $y = \sin x$). If we substitute in $y''+y=0$, $\sin x$ for y and $-\sin x$ for y'' (if $y=\sin x$, then $y' = \cos x$ and $y'' = -\sin x$), we obtain $\sin x - \sin x = 0$ and thus the differential equation is satisfied.

It is easy to show that the equation $y''+y = 0$ is also solved by the function (curve) $y = \cos x$, and also by the function of the type (or the curve representing the equation) $y = C_1 \sin x + C_2 \cos x$, where C_1 and C_2 are any numbers. We have thus established that differential equations may have an infinite number of solutions. We shall come back to this matter since it is a rule which basically applies to all differential equations.

Similarly, we can show that e.g. the solution of the differential equation $xy'+3x-2y = 0$ is formed by the functions (parabolae) of the equation $y = cx^2+3x$ where c is any number.

Let us examine the simplest differential equation of the first order of the type $y' = f(x)$ which we know well from integral calculus. As we know, the equation $y' = f(x)$ can be presented in the equivalent form $\frac{dy}{dx} = f(x)$ or $dy = f(x)\,dx$.

The latter form, being the most convenient, is most frequently used.

The solution of the differential equation $y' = f(x)$ consists in normal integration, i.e., in finding the original function of the given function $f(x)$. Hence, if $y' = f(x)$ then $y = \int f(x)\,dx$ or what is the same thing, $y = \int_a^x f(t)\,dt + C$; a and C are any constants.

Let us take some specific examples.

(1) if $y' = 2x$, then $y = \int 2x\,dx = x^2+C$, where

C is any constant or[5] $y = \int_a^x 2t\,dt = x^2 - a^2 + C_1 = x^2+C$.

(2) if $y' = \cos x$, then $y = \int \cos x\,dx = \sin x + C$.

(3) if $y' = \frac{1}{x+2}$, then $y = \int \frac{1}{x+2}\,dx = \ln|x+2|+C$.

[5] The algebraic sum of the arbitrary constants $-a^2+C_1$ may, obviously, always be replaced by the one arbitrary constant C.

It follows from these examples and from the theory of integral calculus that the solution (integral) of a differential equation of the first order $y' = f(x)$ is not a single function but *a one-parameter family (group) of functions* $y = \phi(x, C)$. It is *a general solution (or general integral) of a differential equation* of the first order.

However, if the required function must satisfy some *initial condition*, e.g., it is known that for the value x_0 of the independent variable the required function has the value y_0, then this initial condition makes it possible to determine the value of the constant C and in this way to obtain the one definite function satisfying the given differential equation. This is the *particular solution (particular integral) of the differential equation.*

Let us take as an example the differential equation $y' = 3x^2$ and let us find the function satisfying this equation together with the initial condition according to which when $x = 2$ then $y = 9$. We can formulate this initial condition so that the graph of the required function passes through the point having as coordinates 2, 9.

The general solution of an equation obtained by integration has the form

$$y = \int 3x^2 dx, \quad \text{i.e.,} \quad y = x^3 + C,$$

where C is any constant. Since when $x = 2$ $y = 9$, by substituting these values in the general solution of the equation we obtain $9 = 8 + C$, hence the constant $C = 1$. The particular solution of the equation is thus the function $y = x^3 + 1$.

In the general theory of differential equations much importance is attached to the so-called assertions of the existence and uniqueness of the solutions of differential equations. Without going into more detail we can say that a differential equation of the nth order may have a general solution (general integral) which is an n-parameter family of functions (curves) $y = \varphi(x, C_1, C_2, \ldots, \ldots, C_n)$, where φ is a differentiable continuous function in a certain region and we can state that when certain fairly general assumptions are satisfied, then through every point of this region passes one of these curves. It should be added that constants of integration appearing in these equations are mutually independent, i.e., their number cannot be reduced by the introduction of other constants.

For example, let us assume that for a differential equation of the first order which can be expressed in the form $y' = f(x, y)$, a sufficient condition for the existence and uniqueness of an integral in a given region D is that the function $f(x, y)$ should be continuous in region D and that the partial derivative f'_y should be continuous.

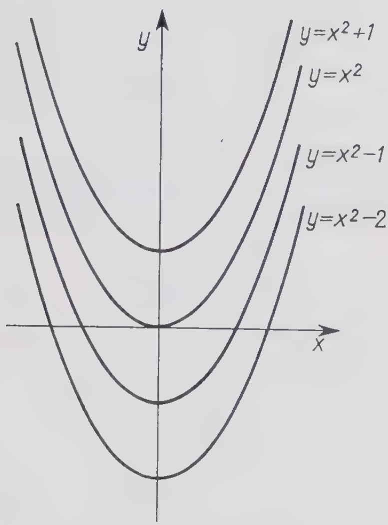

FIG. 18.

The graph of the general integral $y = \varphi\ (x, C)$ of a differential equation of the first order is a one-parameter family of curves which we obtain by drawing graphs of the function $y = \varphi(x, C)$ for different values of any constant C. So, for example, if we have a differential equation $y' = 2x$, the general solution of which has the form $y = x^2+C$, then on the graph the general solution will be a family of parabolae of the type $y = x^2+C$. The particular solution, e.g., satisfying the initial condition $x = 1$, $y = 1$ (and hence $C = 0$), will be a parabola $y = x^2$ passing through the origin of the system of coordinates since only one of the whole family of parabolae $y = x^2+C$ passes through point 1, 1.

3. *Methods of solving differential equations*

We shall now describe some methods of solving simple types of ordinary differential equations and sets of these equations.[6] We shall deal mainly with differential equations of the first order.

(1) Differential equations of the first order with separated variables.

Let us suppose that it is generally possible to solve the differential equation of the first order $F(x, y, y') = 0$ with respect to the derivative function y', i.e., to give to this equation the form

$$y' = -\frac{P(x, y)}{Q(x, y)}, \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are functions of the variables x and y.

We can write equation (1) in the following equivalent forms:

$$\frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)} \tag{1a}$$

or, treating the derivative $y' = \frac{dy}{dx}$, as the ratio of the differentials of the variables y and x:

$$P(x, y)dx + Q(x, y)dy = 0. \tag{1b}$$

There may be special cases in which P is only a function of variable x and Q is a function only of variable y; we can then write the equations (1a) and (1b) in the forms

$$\frac{dy}{dx} = -\frac{f(x)}{g(y)}, \tag{2a}$$

$$f(x)dx + g(y)dy = 0. \tag{2b}$$

Differential equations of the first order which can be set out in the forms (2a) or (2b) are called *differential equations with separable variables.*

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[6] A more detailed discussion of methods of solving differential equations may be found in practically any more comprehensive mathematical textbook on higher mathematics (mathematical analysis) and in special textbooks on the theory of differential equations.

The solution of these equations is very simple. Integrating both sides of the equation (2b) we obtain:[7]

$$\int f(x)\,dx+\int g(y)\,dy = C,$$

and assuming that $\int f(x)\,dx = F(x)$ and $\int g(y)\,dy = G(y)$,[8] we can write the differential equation (2a) or (2b) in the involved form

$$F(x)+G(y) = C$$

and hence determine the unknown function $y = \varphi(x)$.

Example

Solve the differential equation

$$xy'-y = 0.$$

From this equation we determine $y' = \frac{y}{x}$, or directly separate the variables:

$$x\frac{dy}{dx} = y \quad \text{or} \quad \frac{dx}{x}-\frac{dy}{y} = 0.$$

Note that in this case

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(y) = \frac{1}{y}.$$

Integrating the equation we arrive at

$$\int\frac{dx}{x}-\int\frac{dy}{y} = C;$$

[7] We know from integral calculus that if the differentials of two functions $F(x)$ and $G(y)$ equal each other, i.e., $f(x)\,dx = g(y)\,dy$, where $f(x) = F'(x)$ and $g(y) = G'y$ then the functions differ by a constant. It follows from the equation $f(x)\,dx = g(y)\,dy$ that $F(x) = G(y)+C$.

[8] We see that in order to solve a differential equation it is necessary to determine indeterminate integrals which is not always easy and may even be impossible. In cases of this kind approximate solutions are found for differential equations.

hence
$$\ln|x|-\ln|y| = \ln|C|$$
or
$$\ln\left|\frac{x}{y}\right| = \ln|C|.$$

It follows that $\frac{x}{y} = C$, and hence $y = \frac{x}{C}$. This is the general integral of the differential equation $xy'-y = 0$. It is easy to see that the general integral is in this case a pencil of straight lines passing through the origin of the system of coordinates.

The solution may be checked by substituting $y = \frac{x}{C}$ in the differential equation. We then have $x\frac{1}{C}-\frac{x}{C} = 0$, and the equation is thus satisfied.

The method of separating the variables can also be used to solve differential equations of the type

$$P(x, y)dx+Q(x, y)dy = 0, \tag{1b}$$

if $P(x, y)$ and $Q(x, y)$ can be expressed as products of functions of which each is a function only of x or only of y;

$$f_1(x)g_1(y)dx+f_2(x)g_2(y)dy = 0. \tag{3}$$

Dividing both sides of the equation (3) by $f_2(x)\,g_1(y) \neq 0$[9] we obtain

$$\frac{f_1(x)}{f_2(x)}dx+\frac{g_2(y)}{g_1(y)}dy = 0,$$

which is an equation in the form (2b) with separated variables which can be solved by the method given above.

Example

Solve the differential equation

$$(1+x)y\,dx+(1-y)x\,dy = 0.$$

Dividing both sides of the equation by xy we obtain

$$\left(\frac{1}{x}+1\right)dx+\left(\frac{1}{y}-1\right)dy = 0.$$

[9] The case $f_2(x)g_1(y) = 0$, which occurs when $f_2(x) = 0$ or $g_1(y) = 0$, requires additional and separate consideration.

Integrating this equation we arrive at a general integral in the involved form

$$\ln|x|+x+\ln|y|-y = C$$

or

$$\ln|xy|+x-y = C.$$

(2) Let us consider a differential equation of the first order of the type

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right). \tag{4}$$

We arrive at this type of equation if y', as a function of the ratio of the independent variable y and of the dependent variable x, can be derived from the differential equation $F(x, y, y') = 0$.

The method employed to solve equation (4) is the one frequently used in the theory of differential equations, i.e., by introducing a new variable, in this case the variable

$$u = \frac{y}{x}.$$

If $u = \frac{y}{x}$, then $y = ux$, and hence the derivative of function y (treated as the product of the two functions u and x) with respect to variable x equals

$$\frac{dy}{dx} = x\frac{du}{dx}+u.$$

Thus, equation (4) may be written as follows:

$$x\frac{du}{dx}+u = f(u). \tag{4a}$$

This is a differential equation of the type (2b) with separable variables x and u. The equation (4a) may be written

$$x\frac{du}{dx} = f(u)-u$$

or

$$\frac{du}{f(u)-u} = \frac{dx}{x}.$$

Integrating both sides we finally obtain the equation

$$\int \frac{du}{f(u)-u} = \int \frac{dx}{x} + C$$

and hence, (after integrating), we determine the variable auxiliary u and then the variable $y = ux$ as the function of x and any constant C.

Example

Solve the differential equation

$$\frac{dy}{dx} = \frac{xy}{x^2-y^2}.$$

Dividing the numerator and the denominator on the right-hand side of this equation by x^2 we obtain

$$\frac{dy}{dx} = \frac{\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2},$$

hence the right-hand side of equation is a function of the $\frac{y}{x}$. When a new variable $u = \frac{y}{x}$ is introduced, the equation can be written as

$$u + x\frac{du}{dx} = \frac{u}{1-u^2}.$$

In order to solve this equation we separate the variables as shown below:

$$x\frac{du}{dx} = \frac{u}{1-u^2} - u,$$

$$\frac{x}{dx} = \frac{u^3}{1-u^2}\,\frac{1}{du},$$

$$\frac{dx}{x} = \frac{1-u^2}{u^3}\,du,$$

$$\frac{dx}{x} = (u^{-3}-u^{-1})\,du.$$

Integrating both sides we finally arrive at the equation[10]

$$\ln|x|+\ln|C| = -\frac{1}{2u^2}-\ln|u|.$$

Substituting for u its value $\frac{y}{x}$, we obtain, after some simple transformations:

$$-\frac{x^2}{2y^2} = \ln|Cy|.$$

The solution of this differential has a less simple form. To present it in the distinct form: $y = f(x)$ would not be easy.

(3) A linear differential equation of the first order.

As we know, a linear differential equation of the first order can be written in the form

$$\frac{dy}{dx}+p(x)y = q(x). \tag{5}$$

Let us first consider the case when the equation is homogeneous, i.e., when $q(x) = 0$.

In order to solve the equation

$$\frac{dy}{dx}+p(x)y = 0 \tag{5a}$$

we separate the variables:

$$\frac{dy}{y} = -p(x)dx.$$

Integrating the equation we obtain

$$\ln|y|-\ln|C| = -\int p(x)dx$$

and hence

$$\ln\frac{y}{C} = -\int p(x)dx,$$

$$y = Ce^{-\int p(x)dx}.$$

[10] It should be noted that the arbitrary constant may be put in a form more convenient for later transformation; $1n|C|$ (or $-1n|C|$) since for any number C_1 we can choose C such that $C_1 = 1n|C|$.

Non-homogeneous linear differential equations (5) are solved by variation constant, a method often used in the solution of differential equations.[11] To this end we assume that the solution of a non-homogeneous linear differential equation has the same form as the solution of a homogeneous equation with the difference that C is not a constant but is a function of the variable $x: C(x)$.

We then try to choose a function $C(x)$ so that the solution in the form $y = C(x)e^{-\int p(x)dx}$ will satisfy the equation. We obtain:

$$\frac{dC(x)}{dx}e^{-\int p(x)dx}+C(x)e^{-\int p(x)dx}[-p(x)]+p(x)C(x)e^{-\int p(x)dx} = q(x),$$

and after reducing

$$\frac{dC(x)}{dx}e^{-\int p(x)dx} = q(x)$$

or

$$\frac{dC(x)}{dx} = q(x)e^{\int p(x)dx},$$

and hence

$$C(x) = \int q(x)e^{\int p(x)dx}\,dx+A \text{ where } A \text{ is any constant.}$$

By calculating $C(x)$ in this way and substituting it in the solution of the homogeneous equation $y = Ce^{-\int p(x)dx}$ we arrive at the general solution of the non-homogeneous equation.

Example

Solve the differential equation

$$\frac{dy}{dx}-\frac{y}{x} = 3x.$$

---

[11] There are also other ways of solving a linear differential equation of the first order. For example, it can be assumed that the solution of this type of equation has the form of the product of two functions $y = u(x)v(x)$ and assuming that one of these functions, e.g., $u(x)$ is arbitrary, we determine the function $v(x)$ in such a way that the solution $y = u(x)v(x)$ satisfies the equation.

Note that this is a non-homogeneous linear differential equation and that in this case

$$p(x) = -\frac{1}{x} \quad \text{and} \quad q(x) = 3x,$$

hence

$$\int p(x)\,dx = -\int \frac{1}{x}\,dx = -\ln|x|,$$

and because

$$e^{-\ln x|} = \frac{1}{x},$$

hence

$$C(x) = \int 3x \cdot \frac{1}{x}\,dx = 3x + A \text{ where } A \text{ is any constant.}$$

The solution of this differential equation thus takes the form

$$y = C(x)e^{-\int p(x)dx} = (3x+A)e^{\ln|x|}$$

or

$$y = (3x+A)x = 3x^2 + Ax.$$

(4) An incomplete differential equation of the second order.

We shall consider methods of solving the more simple types of differential equations of the second order which—as we already know—can be written in the general form $F(x, y, y', y'') = 0$.

(a) Let us suppose that we have a differential equation of the second order of the type

$$y'' = f(x). \tag{6}$$

This can be simply solved by integrating twice. If $y'' = f(x)$, then $y' = \int f(x)\,dx + C_1$, and hence $y = \int [\int f(x)\,dx + C_1]\,dx + C_2$.

The solution of a differential equation of the second order is a two-parameter family of functions.

Example

If

$$y'' = \sin x, \quad \text{then} \quad y' = -\cos x + C_1 \quad \text{and}$$

$$y = \int (-\cos x + C_1)dx = -\sin x + C_1 x + C_2 .$$

(b) A differential equation of the second order can be solved equally simply when it is of the type

$$y'' = f(y). \tag{7}$$

Let $y' = p$. Treating p as a function of y, which is, in turn, a function of x, we obtain

$$y'' = \frac{dy'}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p \frac{dp}{dy}.$$

Substituting y'', calculated in this way, in the equation $y'' = f(y)$ we obtain

$$p \cdot \frac{dp}{dy} = f(y).$$

This latter equation can be solved by the separation of variables. We have, $p\,dp = f(y)\,dy$, hence, after integrating

$$\frac{p^2}{2} = \int f(y)\,dy + \frac{C_1}{2},$$

or

$$p = y' = \pm \sqrt{2\int f(y)dy + C_1},$$

and hence

$$y = \pm \sqrt{2\int f(y)\,dy + C_1\,dx + C_2}.$$

Example

In order to solve the equation $y'' = y^{-3}$, let

$$y' = p, \quad \text{or} \quad y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} p .$$

The equation $y'' = y^{-3}$ can, then, be transformed as follows:

$$p\frac{dp}{dy} = y^{-3} \quad \text{or} \quad p\,dp = y^{-3}dy.$$

After integrating the latter equation we obtain

$$\frac{p^2}{2} = \frac{y^{-2}}{-2} + \frac{C_1}{2},$$

hence

$$p = \frac{dy}{dx} = \pm\sqrt{C_1 - y^{-2}}.$$

This latter differential equation is of the first order and may be easily solved by separation of variables. It can be shown that the solution of the equation $y'' = y^{-3}$ will have the form

$$C_1 y^2 - 1 = (C_1 x + C_2)^2.$$

(c) Let us now consider the third special case of a differential equation of the second order

$$y'' = f(y'). \tag{8}$$

In order to solve this equation we again denote y' by p; then $y'' = \frac{dp}{dx}$ and the differential equation $y'' = f(y)$ is transformed into the equation $\frac{dp}{dx} = f(p)$, which we solve (with respect to p) by the separation of variables. Knowing $p = \frac{dy}{dx}$, we determine y by integrating again.

Example

In order to solve the equation $2y'y'' = 1$, let $y' = p$. Then the differential equation takes the form

$$2p\frac{dp}{dx} = 1$$

or

$$2p\,dp = dx.$$

Integrating this equation we obtain

$$p^2 = x + C_1,$$

hence
$$p = \pm\sqrt{x+C_1}$$

or
$$\frac{dy}{dx} = \pm\sqrt{x+C_1}.$$

Separating the variables in this last equation and integrating both sides of the equation we obtain

$$y = \pm\int\sqrt{x+C_1}\,dx + C_2 = \pm\frac{2}{3}(x+C_1)^{3/2} + C_2.$$

(5) A linear differential equation of the second order with constant coefficients.

We shall now discuss the principles of solving homogeneous differential equations of the second order with constant coefficients which can be written in the form

$$y'' + py' + qy = 0, \tag{9}$$

where p and q are constants.

The solution of this type of equation depends on the following theorem which we give without proof:

If y_1 and y_2 are linearly independent[12] particular solutions of the equation, (9) then the general solution of this equation takes the form[13]

$$y = C_1 y_1 + C_2 y_2, \quad \text{where } C_1 \text{ and } C_2 \text{ are any constants.}$$

It follows from this theorem that the solution of a linear differential equation of the second order with constant coefficients consists in finding two particular solutions of it which are linearly independent. Let us suppose that these particular solutions of the

[12] It should be remembered that the two variables y_1 and y_2 are termed linearly independent if the identity $C_1 y_1 + C_2 y_2 \equiv 0$ (where C_1 and C_2 are any constants) is satisfied only for $C_1 = 0$ and $C_2 = 0$. If the identity is also satisfied for instance for $C_2 \neq 0$, then $y_2 = -\frac{C_1}{C_2} y_1$, and hence it follows that y_2 is proportional to y_1 and hence y_1 and y_2 are linearly dependent.

[13] That is to say that the general solution y is the weighted sum of the particular solutions y_1 and y_2 where the weights are any constants.

equation (9) have the form $y = e^{kx}$ wherein the constant k must be determined.[14]

Substituting $y = e^{kx}$ in equation (9) we have[15]

$$k^2e^{kx}+pke^{kx}+qe^{kx} = 0$$

or, dividing both sides by $e^{kx} \neq 0$:

$$k^2+pk+q = 0. \tag{10}$$

This is the *characteristic equation of the differential equation.*

The characteristic equation (10) is a quadratic equation with respect to k and hence may have two real roots $\left(\text{if the discriminant of this equation } \Delta = \frac{p^2}{4} - q > 0\right)$, one real double root (if $\Delta = 0$) or two conjugate complex roots (if $\Delta < 0$).

Let us consider these three cases in turn:

(1) If the discriminant of the characteristic equation $\Delta > 0$, then the equation has two roots k_1 and k_2 and two particular solutions of the differential equation under consideration $y''+py'+qy = 0$ are as follows,

$$y_1 = e^{k_1x} \quad \text{and} \quad y_2 = e^{k_2x}$$

and hence—in accordance with the theorem cited above—the general solution of a linear differential equation of the second order with constant coefficients has the form

$$y = C_1e^{k_1x}+C_2e^{k_2x}. \tag{11}$$

(2) If the discriminant of the characteristic equation $\Delta = \frac{p^2}{4} - q = 0$, then the equation has one (double) root $k_1 = k_2 = -\frac{p}{2}$ and in this case we have only one particular solution $y_1 = e^{-\frac{p}{2}x}$.

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[14] It should be noted that the two functions $y_1 = e^{k_1x}$ and $y_2 = e^{k_2x}$ are linearly independent. If these functions were linearly dependent, then $C_1e^{k_1x} + C_2e^{k_2x} \equiv 0$, where at least one of the coefficients, e.g., $C_2 \neq 0$. But then $e^{(k_2-k_1)x} \equiv -\frac{C_1}{C_2}$, which is impossible since the right-hand side is constant (independent of x) and the left-hand side is variable.

[15] If $y = e^{kx}$, then $y' = ke^{kx}$ and $y'' = k^2e^{kx}$.

In order to find y_2, i.e., the second particular solution of the differential equation $y''+py'+qy=0$, let us assume that the solution has the form $y=y_1z$, where $z=z(x)$ is a function of x not identically equal to a constant.

In order to find the function z for which the equation $y_2=y_1z$ satisfies equation (9), we calculate:

$$y_2' = e^{-\frac{p}{2}x}\left(z'-\frac{p}{2}z\right)$$

$$\text{and} \qquad y_2'' = e^{-\frac{p}{2}x}\left(z''-pz'+\frac{p^2}{4}z\right),$$

and then substitute y_2, y_2', y_2'' in equation (9).

After dividing $e^{-\frac{p}{2}x}$ and after reducing we obtain the equation

$$z''+\left(q-\frac{p^2}{4}\right)z=0,$$

and since in this case $q-\frac{p^2}{4}=0$, therefore, also $z''=0$.

From the condition $z''=0$ it follows that $z'=C_2$ and $z=C_2x+C_3$, where C_2 and C_3 are any constants. Because we want to find the particular solution of equation (9), we may assume that

$C_2=1$ and $C_3=0$, and then $z=C_2x$, and $y_2=C_2xe^{-\frac{px}{2}}$.

Thus, in accordance with the theorem mentioned above, the general solution of equation (9), in the case where the discriminant of the characteristic equation is equal to zero, has the form

$$y=e^{-\frac{p}{2}x}(C_1+C_2x). \tag{12}$$

(3) Let us consider the last case where the discriminant of the characteristic equation $\Delta=\frac{p^2}{4}-q<0$; in this case the characteristic equation (10) has two conjugate complex roots which we can write in the form

$$k_1=\alpha+\beta i \quad \text{and} \quad k_2=\alpha-\beta i \quad \text{where} \quad i^2=-1.$$

In this case the particular solutions of equation (9) have the form

$$y_1 = e^{(\alpha+\beta i)x}, \; y_2 = e^{(\alpha-\beta i)x}.$$

Using Euler's formulae,[16] we can transform these equations as follows:

$$y_1 = e^{(\alpha+\beta i)x} = e^{\alpha x}e^{\beta ix} = e^{\alpha x}(\cos\beta x + i\sin\beta x),$$
$$y_2 = e^{(\alpha-\beta i)x} = e^{\alpha x}e^{-\beta ix} = e^{\alpha x}(\cos\beta x - i\sin\beta x).$$

These are complex[17] solutions of the differential equation (9), which make it possible to find easily two other real solutions of this differential equation.

Indeed, it can be shown that if y_1 and y_2 are linearly independent particular solutions of the differential equation (9), then the sum of these solutions divided by 2

$$Y_1 = \frac{y_1+y_2}{2} = e^{\alpha x}\cos\beta x$$

and their difference divided by $2i$

$$Y_2 = \frac{y_1-y_2}{2i} = e^{\alpha x}\cos\beta x,$$

are linearly independent particular solutions of a given differential equation.

If, on the other hand, Y_1 and Y_2 are two linearly independent solutions of the differential equation (9), then their weighted sum

$$y = C_1Y_1 + C_2Y_2 = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x) \tag{13}$$

will be the required general solution of equation (9); C_1 and C_2 are any constants (real or complex).

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[16] Euler's formulae:

$$e^{ix} = \cos x + i\sin x, \quad e^{-ix} = \cos x - i\sin x,$$

hence it follows that

$$\cos x = \frac{e^{ix}+e^{-x}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}.$$

[17] The solutions y_1 and y_2 are complex functions (of the real variable x) since they contain elements with the imaginary factor $i = \sqrt{-1}$.

The appearance of trigonometric functions, i.e. of periodic functions in the general solution of a differential equation in the case of complex roots of the characteristic equation (10) shows that the differential equation under consideration is related to phenomena of a periodic pattern.

In order to solve a non-homogeneous differential equation of the second order with constant coefficients, i.e., an equation in the form

$$y''+py'+qy = v(x), \tag{14}$$

we frequently make use of the following theorem which we give without proof.

The solution of the non-homogeneous equation (14) is the sum of the general solution of the corresponding homogeneous equation (i.e., the equation $y''+py'+qy = 0$) and of a particular solution of the given non-homogeneous equation.

Because we know the method by which a homogeneous equation is solved, the finding of the general solution of a non-homogeneous equation consists in finding its particular solution.

The determination of the particular solution of the equation (14) depends on the nature of the function $v(x)$. Usually the particular solution sought has the same structure as the function $v(x)$. Often, however, it may be possible to find the particular solution of equation (14) by the method of trial and error.

In order to explain this method we shall confine ourselves to one simple example where the right-hand side of the equation (14) is an exponential function, i.e., $v(x) = ae^{mx}$, where $a \neq 0$.

In this case we assume that the particular solution of the equation (14) may also be written as the exponential function $z = Ae^{mx}$, where A is an indeterminate coefficient.

From this assumption it follows that $z' = Ame^{mx}$ and $z'' = Am^2e^{mx}$.

If the function $z = Ae^{mx}$ is the solution of equation (14), then, if it is substituted in this equation for y (and correspondingly z' and z'' for y' and y''), it satisfies the equation. Hence (after dividing both sides of the equation by e^{mx}) we obtain

$$A(m^2+pm+q) = a.$$

Two cases are possible:

(a) m is not a root of the characteristic equation (10) (corresponding to the homogeneous equation $y''+py'+qy = 0$);

(b) m is a root of the characteristic equation (10).

In the first case $m^2+pm+q \neq 0$, and hence $A = \dfrac{a}{m^2+pm+q}$ and the particular solution of the equation (14) can be written

$$z = \frac{ae^{mx}}{m^2+pm+q}.$$

In the second case $m^2+pm+q = 0$, hence the equation $A\ (m^2+pm+q) = a$ is contradictory and equation (3) has no solution in the form $z = Ae^{mx}$.

Because of this we try—as above—to see whether the solution of equation (14) could be expressed in a more complex form, e.g., $z = Axe^{mx}$, $z = Ax^2e^{mx}$ etc.

Example

Solve the equation $y''-5y'+6y = e^x$.

First of all we look for the general solution of the homogeneous equation

$$y''-5y'+6y = 0.$$

Because in this case $p = -5$, $q = 6$ hence the characteristic equation is

$$k^2-5k+6 = 0.$$

This characteristic equation has real roots ($\Delta = 1 > 0$), $k_1 = 2$ and $k_2 = 3$, and therefore the function

$$y = C_1e^{3x}+C_2e^{2x}$$

is the general solution of the equation $y''-5y'+6y = 0$.

We now proceed to look for the particular solution assuming that the solution has the form $z = Ae^x$ (in this case we have taken $m = 1$, which is possible since 1 is not a root of the characteristic equation $k^2-5k+6 = 0$). Substituting

$$z = Ae^x,\ z' = Ae^x,\ z'' = Ae^x$$

for y, y', and y'' respectively in the equation which we have to solve, we obtain

$$Ae^x - 5Ae^x + 6Ae^x = e^x,$$

and after reducing and simplifying we arrive at $2A = 1$, hence $A = 1/2$. The general solution of this non-homogeneous equation will be the function

$$y = C_1 e^{3x} + C_2 e^{2x} + \frac{1}{2} e^x.$$

4. *A system of linear differential equations of the first order*

In order to familiarize ourselves with one typical method of solving a set of differential equations, let us consider the particular case of a set of n linear differential equations of the first order with constant coefficients in the form

$$\begin{cases} \dfrac{dy_1}{dx} = a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n, \\ \dfrac{dy_2}{dx} = a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{dy_n}{dx} = a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n, \end{cases} \tag{1}$$

where $y_1, y_2, \ldots, y_n$ are unknown functions and the coefficients a_{ik} $(i, k = 1, 2, \ldots, n)$ are constants.

It follows from the theorems on the existence of solutions for differential equations that if we know n particular solutions of the set (1) in the form of n sequences of functions

$$\begin{cases} y_{11}, y_{12}, \ldots, y_{1n}, \\ y_{21}, y_{22}, \ldots, y_{2n}, \\ \ldots\ldots\ldots\ldots\ldots \\ y_{n1}, y_{n2}, \ldots, y_{nn}, \end{cases} \tag{2}$$

then the set of n linear combinations of functions

$$\begin{cases} y_1 = C_1 y_{11} + C_2 y_{12} + \ldots + C_n y_{1n}, \\ y_2 = C_1 y_{21} + C_2 y_{22} + \ldots + C_n y_{2n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n = C_1 y_{n1} + C_2 y_{n2} + \ldots + C_n y_{nn}, \end{cases} \tag{3}$$

where $C_1, C_2, \ldots, C_n$ are any constants, is also a solution of the set of differential equations (1).

If, moreover, the determinant

$$\begin{vmatrix} y_{11} y_{12} \cdots y_{1n} \\ y_{21} y_{22} \cdots y_{2n} \\ \cdots\cdots\cdots\cdots \\ y_{n1} y_{n2} \cdots y_{nn} \end{vmatrix} \neq 0, \tag{4}$$

then from the linear equations (3) it is possible to determine the values of the constants $C_1, C_2, \ldots, C_n$, such that the functions $y_1, y_2, \ldots, y_n$ have given values at any point $x = x_0$. In this case, the set of n linear combinations (3) will be the general solution of the set of differential equations (1).

In order to solve the set of differential equations (1) let us assume that the particular solutions of this set form the sequence of n exponential functions:

$$y_1 = \alpha_1 e^{rx},\ y_2 = \alpha_2 e^{rx},\ \ldots,\ y_n = \alpha_n e^{rx}. \tag{5}$$

In order to determine the constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ and the constant r, we substitute these functions in the set (1). We obtain (after dividing both sides of the set of equations by e^{rx} and arranging) the following set of n homogeneous numerical linear equations

$$\begin{cases} (a_{11}-r)\alpha_1 + a_{12}\alpha_2 + \ldots + a_{1n}\alpha_n = 0, \\ a_{21}\alpha_1 + (a_{22}-r)\alpha_2 + \ldots + a_{2n}\alpha_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots + (a_{nn}-r)\alpha_n = 0. \end{cases} \tag{6}$$

This set has a solution other than zero with respect to $\alpha_1, \alpha_2, \ldots, \ldots, \alpha_n$ when and only when the determinant of the coefficients of this set equals zero:

$$W(r) = \begin{vmatrix} a_{11}-r & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-r & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-r \end{vmatrix} = 0. \tag{7}$$

It is easy to see (after expanding the determinant $W(r)$) that condition (7) is an equation of the nth degree with respect to the unknown r. For each root r_i of this equation there are corresponding non-zero solutions $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}$ of the set of linear equations (6). If the equation has n single roots $r_1, r_2, \ldots, r_n$ then there are n sequences of functions

$$\begin{cases} y_{11} = \alpha_{11} e^{r_1 x}, \; y_{12} = \alpha_{12} e^{r_1 x}, \; \ldots, \; y_{1n} = \alpha_{1n} e^{r_1 x}, \\ y_{21} = \alpha_{21} e^{r_2 x}, \; y_{22} = \alpha_{22} e^{r_2 x}, \; \ldots, \; y_{2n} = \alpha_{2n} e^{r_2 x} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_{n1} = \alpha_{n1} e^{r_n x}, \; y_{n2} = \alpha_{n2} e^{r_n x}, \; \ldots, \; y_{nn} = \alpha_{nn} e^{r_n x}, \end{cases} \tag{8}$$

each of which is a solution of the set of differential equations (1). It is possible to prove that the determinant (4) for the functions (8) is not equal to zero and hence the set of linear combinations in the form (3) of solutions (8) provides the general solution of the set of equations (1).

5. *Difference equations*

Differential equations are found in practice (e.g., in the study of physical phenomena, in solving certain economic problems, etc.) when we are dealing with quantities and functions which are continuous. If, on the other hand, it appears from the nature of a particular phenomenon that function y changes in steps, then the study of phenomena of this type by mathematical methods leads in practice to *difference equations*.

It often happens that the same phenomenon may be considered both as a continuous process and as a step process. For example, the growth of the national income, a change in the investment fund etc., may be considered as continuous processes in the sense that they are continuously increasing at every moment in the period studied. It is also possible—and this is sometimes more convenient—to assume that the size of the national income, the investment fund and other similar quantities change by steps, e.g., at the end of every year, quarter, month, etc.

As a specific example let us consider the case where y changes in relation to time x "according to the principle of the compound rate of interest". In treating this process as a discrete one let us assume that the original value of the dependent variable is y_0, the compound unit rate of interest is r,[18] and the period after which the quantity considered increases at the given rate is one year. We shall take this period as the unit of time measurement, i.e. of the quantity x.

We then have the following relation

$$y_x = (1+r)y_{x-1}, \tag{1}$$

which is a simple difference equation determining the quantity y_x—corresponding to the determined level of the dependent variable x—depending on the quantity y_{x-1} corresponding to the level of the dependent variable one period earlier.[19]

Since the relation (1) holds for $x = 1, 2, 3, \ldots$, it is easy to check (by the repeated application of formula (1)), that the solution of the difference equation (1) may be written

$$y_x = y_0(1+r)^x \quad (x = 1, 2, 3, \ldots). \tag{2}$$

Let us now consider a process of growth at a compound rate of interest, as a continuous process. For this purpose let us initially assume that interest is added to the value of variable y not at the end of each unit of time (e.g., each year), but at periods equal to $\frac{1}{n}$ of this unit. The new unit rate of interest, moreover, for each period of $\frac{1}{n}$ of the unit of time will equal $\frac{r}{n}$, i.e., $\frac{1}{n}$ of the rate r for the unit of time.

The relation (1) can then be written

$$y_{x+\frac{1}{n}} = \left(1+\frac{r}{n}\right)y_x \tag{3}$$

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[18] The growth per unit in a given period of time is called the unit rate of interest. If the rate of interest is, e.g., 5 per cent, then the unit rate of interest is 5/100. In economic textbooks the unit rate of interest is frequently denoted by the letter r, while in textbooks on political arithmetic the unit rate of interest is denoted by the letter i. Letter r is the interest factor, $r = 1+i$.

[19] The symbol y_x (or y_{x-1}) is a simplified way of writing the value of function y corresponding to the independent variable equal to x (or $x-1$). The symbol y_x is thus interchangeable with the symbols $y(x)$ or $f(x)$.

and will hold for

$$x = 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots$$

It follows from equation (3) that

$$\frac{y_{x+\frac{1}{n}} - y_x}{\frac{1}{n}} = r y_x$$

or

$$\frac{y_{x+h} - y_x}{h} = r y_x, \quad \text{where} \quad h = \frac{1}{n}. \tag{3a}$$

The assumption that y changes continuously is the same as assuming that the interest rate period $\frac{1}{n}$ becomes shorter and shorter, i.e., $n \to \infty$ and $h = \frac{1}{n} = \to 0$.

Note, too, that when $h \to 0$, then the left side of the equation (3a) is equal to the derivative of the function y with respect to the variable x and the equation (3a) is transformed into the differential equation

$$\frac{dy_x}{dx} = r y_x. \tag{4}$$

By separating the variables and taking into account the condition that when $x = 0$ then $y = y_0$, it is easy to find that the solution of equation (4) is the exponential function

$$y_x = y_0 e^{rx}. \tag{5}$$

This is the well-known formula for continuous compound interest, i.e., a formula which makes it possible to determine the value of y_x in relation to the time x, assuming that y_x changes continuously at the compound rate of interest.

The basic concepts in the theory of difference equations are *differences of functions*. Let us assume that the independent variable x of a given function $y=f(x)$ assumes the value $x=x_0$, $x_1, x_2, \ldots, x_n$, and the intervals between these consecutive values of the independent variable x (*difference intervals*) are equal, i.e.,

$$x_1-x_0 = x_2-x_1 = \ldots = x_n-x_{n-1} = h.$$

Without making this assumption less general, we may assume that the difference interval $h = 1$. If, for example, in a given dynamic process the independent variable x is time, and the difference between consecutive values of the variable x is one month, then there is nothing to prevent the adoption of this period as the unit of time measurement. In view of this, the sequence of consecutive values of the independent variable x is

$$x,\ x+1,\ x+2,\ \ldots,\ x+n,$$

and the sequence of the values of the function $y = f(x)$ corresponding to the values of x given above can be written

$$y_x,\ y_{x+1},\ y_{x+2},\ \ldots,\ y_{x+n}.$$

The new function defined by the formula[20]

$$\Delta y_x = y_{x+1}-y_x \tag{6}$$

is called *the first difference of the function* $y=f(x)$.

From this definition it follows that the first difference of the function $y=f(x)$ for consecutive values of the variable x assumes the following values

$$\Delta y_x = y_{x+1}-y_x,\ \Delta y_{x+1} = y_{x+2}-y_{x+1},\ \Delta y_{x+2} = y_{x+3}-y_{x+2},\ \ldots$$

Similarly we define the second, third and further differences of the function, i.e., we form the second difference $\Delta^2 y_x$ as the first difference of the function which is itself the first difference of the function y_x. Hence, we obtain

$$\Delta^2 y_x = \Delta(\Delta y_x) = \Delta y_{x+1}-\Delta y_x, \tag{6a}$$

similarly

$$\Delta^3 y_x = \Delta^2 y_{x+1}-\Delta^2 y_x \tag{6b}$$

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[20] The symbol Δ is here the symbol for the difference operator which transforms the given function y_x into the new function called its first difference, just as the symbol D introduced by Cauchy is the operator transforming the given function $y = f(x)$ into its derivative function $Dy = y'$.

and in general

$$\Delta^n y_x = \Delta^{n-1} y_{x+1} - \Delta^{n-1} y_x. \tag{6c}$$

The way in which consecutive differences of a function are determined can be explained by giving examples.

(1) The first difference of the linear function $y = ax+b$ is constant and equal to a. Indeed, by definition (6) for each value of x we have

$$\Delta y_x = y_{x+1} - y_x = [a(x+1)+b]-(ax+b) = a.$$

It follows that on the basis of formula (6a) $\Delta^2 y_x = 0$.

(2) Let us calculate the consecutive differences of the function $y = x^3$:

$$\Delta y_x = y_{x+1} - y_x = (x+1)^3 - x^3 = 3x^2+3x+1,$$

$$\Delta^2 y_x = \Delta y_{x+1} - \Delta y_x = [3(x+1)^2+3(x+1)+1] - [3x^2+3x+1] = 6x+6.$$

$$\Delta^3 y_x = \Delta^2 y_{x+1} - \Delta^2 y_x = [6(x+1)+6]-[6x+6] = 6.$$

Since the function $\Delta^3 y_x$ is constant, the fourth and higher differences of the function $y = x^3$ are equal to zero.

As an illustration of this example, we give below a table of the values of the function $y = x^3$ and of its differences Δy, $\Delta^2 y$, $\Delta^3 y$ and $\Delta^4 y$ for $x = 0, 1, 2, 3$.[21]

x	x^3	Δx^3	$\Delta^2 x^3$	$\Delta^3 x^3$	$\Delta^4 x^4$
0	0	1	6	6	0
1	1	7	12	6	0
2	8	19	18	6	0
3	27	37	24	6	0

It should be noted that (when $\Delta x = 1$), there is a similarity between the concept of differences of functions and the concept of derivatives of functions with which we deal in differential calculus.

[21] In the symbols for the differences of functions Δy_x, $\Delta^2 y_x$ and so on, there is no need to denote the independent variable x provided that this does not lead to misunderstandings. Similarly, the symbol of the dependent variable y may be replaced by a formula for the function, by writing e.g. Δx^3, $\Delta \sin x$ etc.

However, the use of derivatives in mathematics is limited since the necessary condition for the existence of a derivative is the continuity of the function. On the other hand, differences of functions also exist for non-continuous functions, e.g., those defined by a set of integers.

Just as methods for calculating derivatives of functions of various types are given in differential calculus so there is a *difference calculus* which establishes the rules and formulae for calculating differences of functions. We give some of these formulae[22] without proof (which are, in any case, quite straightforward and are based on the definition of differences).

(1) $\Delta(cy) = c\Delta y$ (the constant may be placed in front of the symbol for the operator Δ).

(2) $\Delta(y_1+y_2) = \Delta y_1+\Delta y_2$.

This is the formula for the difference of the sum of two functions y_1 and y_2 which can easily be generalized to apply to the sum of a finite number of functions. We then arrive at

$$\Delta\sum_{k=1}^{n} C_k y_k = \sum_{k=1}^{n} C_k \Delta y_k.$$

(3) $\Delta[u(x)v(x)] = u(x+1)\Delta v(x)+v(x)\Delta u(x)$,

(4) $\Delta\dfrac{u(x)}{v(x)} = \dfrac{v(x)\Delta u(x)-u(x)\Delta v(x)}{v(x)\cdot v(x+1)}$ (if $v(x)v(x+1) \neq 0$).

The last two formulae are used for calculating the differences of the product and of the ratio of the two functions.

Example

$$\Delta x^4 = \Delta(x\cdot x^3) = (x+1)\Delta x^3+x^3\Delta x$$
$$= (x+1)(3x^2+3x+1)+x^3 = 4x^3+6x^2+4x+1.$$[23]

We shall now give a definition for, and some simple examples of, difference equations.

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[22] The reader's attention is drawn to the similarity between some formulae for calculating differences of functions and derivatives of functions.

[23] We have made use here of the formula given above

$$\Delta x^3 = 3x^2+3x+1.$$

The name *difference equation* is given to an equation expressing the relation between an independent variable x, and an unknown function (dependent variable) y, and its differences $\Delta y, \Delta^2 y, \ldots, \ldots, \Delta^n y$.

According to this definition, we can write a difference equation symbolically in the form

$$\Phi(x, y, \Delta y, \Delta^2 y, \ldots, \Delta^n y) = 0. \quad (7)$$

A difference equation may be defined for all real values of x and for any difference interval h. Most frequently, however, difference equations are defined for a special set of values of x, i.e., for a finite or infinite set of consecutive integers:

$$x_0, x_0+1, x_0+2, x_0+3, \ldots$$

($x_0 = 0$ is frequently taken as the initial level).

In order to indicate that a given function y appearing in a difference equation is defined by a set of consecutive integers $k = 1, 2, 3 \ldots$, it is usual practice to add to the symbol for the function y the letter k writing y_k (instead of y_x, used in the general case).

It can be shown that every difference equation in form (7) may be transformed into form

$$F(x, y_1, y_2, \ldots, y_n) = 0 \quad (7a)$$

expressing the relation between the independent variable x and the values of the function y at various points in the set by which the difference equation is defined.

There are formulae which help in the transformation of a difference equation from (7) into form (7a). In simple cases, however, this transformation does not present any great difficulty as the following examples show.

(1) $\Delta y_k + 3y_k = 0$ is a difference equation of form (7).
Considering that, according to the definition of the first difference of the function $\Delta y_k = y_{k+1} - y_k$, this equation may be written $y_{k+1} + 2y_k = 0$.

(2) $\Delta^2 y_k + 2\Delta y_k + y_k = 0$ is a difference equation.
Since $\Delta y_k = y_{k+1} - y_k$ and

$$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k = (y_{k+2} - y_{k+1}) - (y_{k+1} - y_k) = y_{k+2} - 2y_{k+1} + y_k,$$

it may be written

$$y_{k+2}-2y_{k+1}+y_k+2y_{k+1}-2y_k+y_k = 0;$$

hence

$$y_{k+2} = 0.$$

A difference equation is *linear* if it can be written in the form

$$a_0(k)y_{k+n}+a_1(k)y_{k+n-1}+ \ldots +a_n(k)y_k = b(k), \tag{8}$$

where $a_0(k)$, $a_1(k)$, ..., $a_n(k)$ and $b(k)$ are functions of k determined for all the values of k of a particular set Z by which the difference equation is defined.

If at the same time $a_0(k)$ and $a_n(k)$ are not equal to zero at every point of the set Z then the equation (8) is of the nth order.

If the functions $a_0(k)$, $a_1(k)$, ..., $a_n(k)$ are constant numbers (i.e., not dependent on k) then the equation (8) is called a *difference equation with constant coefficients*.

For example, the difference equation $y_{k+2}+5y_{k+1}-7y_k = 2k$ is of the second order and the equation $y_{k+2}+5y_{k+1} = 2k$ is of the first order, while both are linear equations with constant coefficients. On the other hand in the linear equation $ky_{k+2}+2y_{k+1}-6y_k = 0$ not all the coefficients are constant and the equation is of the second order only when the set of values by which the equation is defined does not include $k = 0$.

The function y is the *solution of the difference equation* defined by the set of values Z if the values of this function defined by set Z satisfy the equation.

We shall not concern ourselves at this point with a discussion of the methods of solving difference equations[24] (which are often similar to those employed in solving differential equations), but shall confine ourselves to giving some simple examples of solutions of equations of this kind.

Given the difference equation $y_{k+1}-2y_k = 0$, determined for $k = 0, 1, 2, 3, \ldots$, it is easy to show that the function $y_k = 2^k (k = 0, 1, 2, \ldots)$ is the solution of this equation. Substituting

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[24] A more extensive discussion of methods of solving difference equations can be found in *Introduction to Difference Equations*, by Samuel Goldberg, J. Wiley and Sons, New York–London, 1958, from which some of the examples given here have been taken.

this function in the left-hand side of the equation $y_{k+1}-2y_k = 0$ we obtain the expression $2^{k+1}-2\cdot 2^k$, which is identically (and hence for every $k = 0, 1, 2, \ldots$) equal to zero.

It is found, however, that the function $y_k = 2^k$ is the *particular solution* of the equation in question. For—as is easy to show—there is a family of functions $y_k = C2^k$ (where C is any constant), which satisfies this equation. Indeed, the expression $C2^{k+1}-2C2^k$ is also identically equal to zero.

The solution $y_k = C2^k (k = 0, 1, 2, \ldots)$ is called the *general solution* of the difference equation.

Similarly, it can be shown that the function $y_k = 2^k(C_1+C_2k)$, $k = 0, 1, 2, \ldots$, where C_1 and C_2 are any constants, is the general solution of a linear difference equation of the second order with constant coefficients:

$$y_{k+2}-4y_{k+1}+4y_k = 0.$$

Substituting the function $y_k = 2^k(C_1+C_2k)$ in this equation we obtain the equation

$$2^{k+2}[C_1+C_2(k+2)]-4\cdot 2^{k+1}[C_1+C_2(k+1)] \\ +4\cdot 2^k(C_1+C_2k) = 0,$$

which—as can easily be seen (after dividing by $2^{k+2} = 2\cdot 2^{k+1} = 4\cdot 2^k$ and reducing)—is satisfied identically.

Given the initial conditions which the solution of the above equation must satisfy, e.g., $y_0 = 1$ and $y_1 = 6$, then the values of the constants C_1 and C_2 can be determined with the help of the general solution.

Substituting in the general solution $y_k = 2^k(C_1+C_2k)$, the value $k = 0$ and then $k = 1$, we obtain $y_0 = C_1$ and $y_1 = 2(C_1+C_2)$.

According to the initial conditions

$$y_0 = C_1 = 1,$$

and

$$y_1 = 2(C_1+C_2) = 6.$$

Hence it is easy to determine: $C_1 = 1$ and $C_2 = 2$.

Thus, the particular solution of a difference equation which satisfies the initial conditions $y_0 = 1$ and $y_1 = 6$ can be written as follows:

$$y_k = 2^k(1+2k).$$

In conclusion of our general comments on the theory of difference equations we give a method of solving difference equations, using a simple linear difference equation of the first order with constant coefficients as an example:

$$y_{k+1} = Ay_k + B \qquad (k = 0, 1, 2, 3, \ldots; A \neq 0). \tag{9}$$

In order to solve this equation let us assume that the value of the function y for $x = 0$, i.e., the initial value of the function y_0 is given.

Assuming $k = 0$, we obtain on the basis of equation (9),

$$y_1 = Ay_0 + B,$$

while for $k = 1$

$$y_2 = Ay_1 + B = A(Ay_0 + B) + B = A^2y_0 + B(1+A).$$

Assuming that $k = 3$, we determine

$$y_3 = Ay_2 + B = A[A^2y_0 + B(1+A)] + B = A^3y_0 + B(1+A+A^2).$$

By the method of induction it can be shown that generally

$$y_k = A^k y_0 + B(1+A+A^2+ \ldots +A^{k-1}) \quad \text{for} \quad k = 1, 2, 3, \ldots$$

Since the sum of the geometric progression

$$1+A+A^2+ \ldots +A^{k-1} = \begin{cases} \dfrac{1-A^k}{1-A} & \text{for} \quad A \neq 1, \\ k & \text{for} \quad A = 1, \end{cases}$$

the particular solution of equation (9) can be written as

$$y_k = \begin{cases} A^k y_0 + B\dfrac{1-A^k}{1-A} & \text{for } A \neq 1 \\ y_0 + Bk & \text{for } A = 1 \end{cases} \qquad (k = 0, 1, 2, \ldots). \tag{10}$$

It can also be shown that the general solution of equation (9) will be:

$$y_k = \begin{cases} CA^k + B\dfrac{1-A^k}{1-A} & \text{for } A \neq 1 \\ C + Bk & \text{for } A = 1 \end{cases} \qquad (k = 0, 1, 2, \ldots), \tag{10a}$$

where C is any constant.

E.g., the solution of the equation $y_{k+1} = 2y_k + 1 (k = 0, 1, 2, \ldots)$ with the initial condition $y_0 = 5$, can, on the basis of (10) and because $A = 2$ and $B = 1$, be written as follows:

$$y_k = 5 \cdot 2^k + 1 \frac{1-2^k}{1-2}$$

or

$$y_k = 6 \cdot 2^k - 1 \qquad (k = 0, 1, 2, \ldots).$$

It is easy to calculate that consecutive values of the function which is a solution of the equation $y_{k+1} = 2y_k + 1$ form the following numerical sequence: 5, 11, 23, 47, 95, ...

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