Real Analysis
By [REDCATED]

Introduction

Continuous at a**:** $\forall\epsilon > 0, \exists\delta > 0$ such that $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.
Supremum: The lest upper bound. The unique real number M such that:

- $a \leq M$ for all $a \in A$ and
- If M' is any real number such that $a \leq M'$ for all $a \in A$, then $M \leq M'$

Convergence: a_n converges to a if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - a| < \epsilon$.
Cauchy Sequence: If for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|a_m - a_n| < \epsilon$.
Note: Convergence says that the numbers are getting closer and closer to a as n gets bigger, while Cauchy says that the numbers are getting closer and closer to each other as n gets bigger.
Bolzano-Weierstrass Theorem: Every sequence of real numbers which is bounded must have a convergent subsequence.
Ratio Test: If a sequence satisfies $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = r < 1$, then the series $\sum_n a_n$ converges. If $r > 1$ then the series diverges, if $r = 1$ it is inconclusive.
Intermediate Value Theorem: For a continuous function $f : \mathbb{R} \to \mathbb{R}$, IVT says that for every value c strictly between $f(a)$ and $f(b)$ there is some $x \in [a, b]$ such that $f(x) = c$.
Extreme Value Theorem: For a continuous function $f : \mathbb{R} \to \mathbb{R}$, there are values c and d in $[a, b]$ at which f takes on the extreme values m and M where $m \leq f(x) \leq M$ for all $x \in [a, b]$. Meaning $f(c) = m$ and $f(d) = M$.
Mean Value Theorem: For a function $f : \mathbb{R} \to \mathbb{R}$, continuous on a closed interval $[a, b]$ and differentiable on an open interval (a, b), then there exists a point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$
Rolles Theorem: If f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.
Chain Rule: Given a composition of differentiable functions $\phi(x) = L \circ E(x)$ we have $\phi'(x) = L'(E(x))E'(x)$
Cauchy-Schwarz Inequality:

$$x_1y_1 + ... + a_nb_n \leq \sqrt{a_1^2 + ... + a_n^2}\sqrt{b_1^2 + ... + b_n^2}$$

Sequences and Series of Functions:

Pointwise Convergence: The sequence (f_n) converges pointwise to the function f, iff for every x in the domain we have

$$\lim_{n\to\infty} f_n(x) = f(x)$$

Uniform Convergence: The sequence (f_n) converges uniformly on a set E with limit f if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in E$:

$$|f_n(x) - f(x)| < \epsilon$$

If f_n converges uniformly then it converges pointwise.
Weierstrass M-test: Suppose there exists a sequence $(M_n)_{n\in\mathbb{N}}$ such that $|f_n(x)| \leq M_n$, $\forall x \in E$ and $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} M_n < \infty$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on E.
Lemma: Uniform convergence iff:

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$$

Lemma: If f_n is continuous on some interval, and f isn't, then the convergence isn't uniform.
Uniformly Cauchy: A sequence of functions f_n is uniformly Cauchy if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ then $\forall x \in E$ we have $|f_m(x) - f_n(x)| < \epsilon$
Power Series: is a series of the form

$$\sum_{n=0}^{\infty} a_n(x - c)^n$$

Where a_n are the coefficients and c is its centre.
Radius of Convergence: R is defined by

$$R = \sup\{r \geq 0 : (a_nr^n) \text{ is bounded}\}$$

Unless (a_nr^n) is bounded for all $r \geq 0$, in which case we declare $R = \infty$
It is a general fact that the radius of convergence of a power series is given by:

$$\lim_{n\to\infty} \inf_{k\geq n} |a_k|^{-1/k}$$

Root Test: Let

$$L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

If $L < 1$ then the series converges absolutely. If $L > 1$ then it diverges. if $L = 1$ and the limit approaches strictly from above then the series diverges.

Theorem 2: Assume that $R > 0$. suppose that $0 < r < R$. Then the power series converges uniformly and absolutely on $|x - c| \leq r$ to a continuous function f, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

defines a continous function $f : (c - R, c + R) \to \mathbb{R}$.
Lemma: The two power series

$$\sum_{n=1}^{\infty} a_n(x - c)^n \text{ and } \sum_{n=1}^{\infty} na_n(x - c)^{n-1}$$

have the same radius of convergence.
Theorem 3: $f(z)$ defined above is infinitely differentiable on $|x - c| < R$ where R is the radius of convergence, and for such x:

$$f'(x) = \sum_{n=0}^{\infty} na_n(x - c)^{n-1}$$

and the series converges absolutely and uniformly on $[c-r, c+r]$ for any $r < R$. Moreover:

$$a_n = \frac{f^{(n)}(c)}{n!}$$

Uniformly Continuous: Let I be an interval and let $f : I \to \mathbb{R}$ be a function. We say that f is uniformly continuous on I if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in I$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.
Lemma: Let I be an open interval in $\mathbb{R}$. Suppose $f : I \to \mathbb{R}$ is differentiable and its derivative f' is bounded on I. Then f is uniformly continuous on I.
Proof: Suppose that $|f'(\zeta)| \leq M$ for all $\zeta \in I$. By MVT we have $f(x) - f(y) = (x - y)f'(\zeta)$ for some ζ between x and y. So $|f(x) - f(y)| = |x - y||f'(\zeta)| \leq M|x - y|$. Let $\epsilon > 0$ and let $\delta = \epsilon/M$. If now $|x - y| < \delta$ we have $M|x - y| < M\delta = \epsilon$.
Theorem: Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then it is uniformly continuous.
Proof: Assume f isn't unif cont. Then there is $\epsilon > 0$ and there are sequences x_n, y_n with $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \epsilon$. Bolzano-Weierstrass tells us that (x_n) has a convergent subsequences $x_{n_k} \to x \in [a, b]$. Since $|x_n - y_n| \to 0$, y_{n_k} also converges to x as $k \to \infty$. Continuity of f at x gives that $\lim_{k\to\infty} f(x_{n_k}) = f(x)$ and similarly $\lim_{k\to\infty} f(y_{n_k}) = f(x)$. So $\lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| = 0$. But this contradicts the fact that $|f(x_n) - f(y_n)| \geq \epsilon$ for all n.
Series Converge Pointwise: Series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to a function $s : E \to \mathbb{R}$ on E iff for every $x \in E$

and $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for $k \geq N$ we have $|\sum_{n=1}^k f_n(x) - s(x)| < \epsilon$.

Series Converge Uniform: Series $\sum_{n=1}^\infty f_n(x)$ converges uniformly to $s : E \to \mathbb{R}$ on E iff for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for $k \geq N$ and all $x \in E$ we have $|\sum_{n=1}^k f_n(x) - s(x)| < \epsilon$. Alternatively iff for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for $k \geq N$ we have $\sup_{x\in E} |\sum_{n=1}^k f_n(x) - s(x)| < \epsilon$

Sequence and Series Examples:

- From FPM: $nx^n \to 0$ as $n \to \infty$
- $f : I \to \mathbb{R}$ is uniformly continuous on I iff whenever $s_n, t_n \in I$ are such that $|s_n - t_n| \to 0$, then $|f(s_n) - f(t_n)| \to 0$.
  Proof: Suppose $f : I \to \mathbb{R}$ is unif cont on I and that $s_n, t_n \in I$ are such that $|s_n - t_n| \to 0$ as $n \to \infty$. Let $\epsilon > 0$. By unif cont of f there is a $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Since $|s_n - t_n| \to 0$ there is an N such that $n \geq N$ implies $|s_n - t_n| < \delta$. So if $n \geq N$ we have $|f(s_n) - f(t_n)| < \epsilon$. Now suppose that f is continuous but not unif cont. So there is an $\epsilon > 0$ such that taking $\delta = 1/n$ there are $s_n, t_n \in I$ with $|s_n - t_n|\delta$ but $|f(s_n) - f(t_n)| \geq \epsilon$. So $|s_n - t_n| \to 0$ but $|f(s_n) - f(t_n)|$ doesn't tend to zero.
- Let $f(x) = x^2$ when x is rational and $f(x) = 0$ when x is irrational. f is discontinuous everywhere except zero, since if $\epsilon > 0$ then $|x| < \sqrt{\epsilon}$ implies $|f(x) = f(0)| < \epsilon$. Therefore it can't be differentiable at any point except possibly zero (It is differentiable at zero).
- Radius of convergence of $\sum_{n=0}^\infty a_n^2 x^n$: the sequence is $(a_n r^n)_{n=0}^\infty$ is bounded for $r < R$ and unbounded for $r > R$. So $(a_n^2 r^{2n})$ is bounded for $r^2 < R^2$ and unbounded for $r^2 > R^2$ so that $(a_n^2 s^n)$ is bounded for $s < R^2$ and unbounded for $s > R^2$. So R^2 is the radius of convergence.
- Radius of convergence of $\sum_{n=0}^\infty a_{2n} x^{2n}$ cannot be determined only from R. It might happen that $a_{2n} = 0$ and the radius of convergence is infinite, or the radius of convergence could be R. It could be any value in $[R, \infty)$.
- If R is the radius of convergence for $\sum_{n=0}^\infty a_n x^n$ then the radius of convergence for $\sum_{n=0}^\infty a_n x^{2n}$ is $\rho = \sqrt{R}$ and $\sum_{n=0}^\infty a_n^2 x^n$ is $\rho = R^2$
- If f_n is continuous and converges uniformly to f on J then f is continuous. Proof: Let $a \in J$ and let $\epsilon > 0$. There exists N such that $n \geq N$ and $\forall x \in J$ we have $|f_n(x) - f(x)| < \epsilon/3$. Continuity of f_N at a implies that $\exists \delta > 0$ such that for $|x - a| < \delta$ we have $|f_N(x) - f_N(a)| < \epsilon/3$. So for $|x - a| < \delta$ we have:
  $$|f(x) - f(a)| \leq$$
  $$|f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)| < 3\epsilon/3$$
  So f is continuous at a.
- If f_n is uniformly continuous and converges uniformly to f on J then f is uniformly continuous. Proof: consider any $\epsilon > 0$. There exists N such that $n \geq N$ and $\forall x \in J$ we have $|f_n(x) - f(x)| < \epsilon/3$. Uniform continuity of f_N implies that $\exists \delta > 0$ such that for $x, y \in J, |x-y| < \delta \implies |f_N(x) - f_N(y)| < \epsilon/3$. So for any $x, y \in J$, $|x - y| < \delta$ we have:
  $$|f(x) - f(y)| \leq$$
  $$|f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < 3\epsilon/3$$
  So f is uniformly continuous on J.
- Let $f_n(x) = nx(1 - x^2)^n$ for $0 \leq x \leq 1$. Find limit function and if uniform.
  Proof: If $x = 0, 1$ then $f_n(0) = 0$. If $0 < x < 1$ then $0 < 1 - x^2 < 1$ and so $n(1 - x^2)^n \to 0$. So f_n converges pointiwse to the zero function. If it was uniform convergence we'd have $\int_0^1 f_n \to \int_0^1 0 = 0$. But $\int_0^1 f_n = \frac{n}{2(n+1)} \to 1/2 \neq 0$. So convergence isn't uniform on $[0, 1]$. If $a \leq x \leq 1$ then $1 - x^2 \leq 1 - a^2$ so that $|f_n(x)| \leq n(1-a^2)^n \to 0$ since $a > 0$. So the convergence is uniform on such intervals.
- Is $f(x) = 1/x$ on $(0, \infty)$ uniformly continuous? No.
  Proof: Take $\epsilon = 1$. Consider the sequences $x_n = 1/n$ and $y_n = 1/(n+1)$. Then $|f(x_n) - f(y_n)| = 1$ so there is no $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$.
  What about $[a, \infty)$ for $a > 0$? Yes.
  Proof: Let $\epsilon > 0$. Consider $|f(x) - f(y)|$ for $a \leq x, y$. This equals $|x - y|/|xy| \leq a^{-2}|x - y|$ for such x, y. So if $a > 0$, we take $\delta < \epsilon a^2$ we have $|x - y| < \delta \implies |f(x) - f(y)| < a^{-2}\epsilon a^2 = \epsilon$.
- Let f_n converge uniformly to f. Let (x_n) be a sequence of real numbers which converge to $x \in \mathbb{R}$. Show $f_n(x_n) \to f(x)$.
  Proof: From triangle inequality $|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$. Since it's implied that f is continuous, let $\epsilon > 0$. Then by uniform convergence of f_n to f, there exists $N \in \mathbb{N}$ such that $n \geq N$ we have $|f_n(y) - f(y)| < \epsilon/2$ for all y. Let $y = x_n$. So if $n \geq N$ we have $|f_n(x_n) - f(x_n)| < \epsilon/2$. Since $x_n \to x$, and f is continuous, there is an $M \in \mathbb{N}$ such that $n \geq M$ implies $|f_n(x_n) - f(x_n)| < \epsilon/2$. So if we take $n \geq \max\{N, M\}$ we have: $|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon$
- Proof $\sum_{n=1}^\infty \frac{1}{n^p}$ converges for $p > 1$:
  From integral test we want f continuous, positive, decreasing on $[1, \infty)$ such that $a_n = f(n)$.
  $$\sum_{n=1}^\infty a_n \text{ converges} \iff \int_1^\infty f(x)dx \text{ converges}$$
  So define $f(x) = \frac{1}{x}$, then $\int_1^\infty f(x)dx = [\ln(x)]_1^\infty = \lim_{n\to\infty} \ln(u) - 0 = \infty$.
  So $\sum_{n=1}^\infty \frac{1}{n}$ doesn't converge.

Integration of Real Functions

Characteristic Function: With $E \subseteq \mathbb{R}$ define $\chi_E : \mathbb{R} \to \mathbb{R}$ by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. Let I be a bounded interval, and:

$$\int \chi_I := \text{length}(I) = |I| = b - a$$

.

Step Function: We say $\phi : \mathbb{R} \to \mathbb{R}$ is a step function with respect to $\{x_0, x_1, ..., x_n\}$ if there exists $x_0 < x_1 < ... < x_n$ for some $n \in \mathbb{N}$ such that:

- $\phi(x) = 0$ for $x < x_0$ and $x > x_n$
- ϕ is constant on (x_{j-1}, x_j) $1 \leq j \leq n$

In other words, ϕ is a step function with respect to $\{x_0, x_1, ..., x_n\}$ iff there exists $c_0, c_1, ..., c_n$ such that

$$\phi(x) = \sum_{j=1}^n c_j \chi_{(x_{j-1}, x_j)}(x)$$

Integrals of Step Functions: If ϕ is a step function with respect to $\{x_0, x_1, ..., x_n\}$ which takes the value on c_j on (x_{j-1}, x_j), then

$$\int \phi := \sum_{j=1}^n c_j (x_j - x_{j-1})$$

and

$$\int (\alpha\phi + \beta\psi) = \alpha \int \phi + \beta \int \psi$$

Riemann Integrable Def 1: Let $f : \mathbb{R} \to \mathbb{R}$ we say that f is Riemann Integrable if for every $\epsilon > 0$ there exists step functions ϕ and ψ such that:

- $\phi \leq f \leq \psi$
- $\int \psi - \int \phi < \epsilon$

Theorem 1: A function $f : \mathbb{R} \to \mathbb{R}$ is R-I iff:
$\sup\{\int \phi \; : \; \phi \text{ is a step function and } \phi \leq f\} = \inf\{\int \psi \; : \; \psi \text{ is a step function and } \psi \geq f\}$

Integral Definition: if f is R-I, and ϕ, ψ are step functions, then we define $\int f$:

$$\int f := \sup\{\int \phi : \phi \leq f\} = \inf\{\int \psi : \psi \geq f\}$$

Riemann Integrable Def 2: A function $f : \mathbb{R} \to \mathbb{R}$ is R-I iff there exists sequences of step functions ϕ_n and ψ_n such that:

$$\phi_n \leq f \leq \psi_n \text{ for all } n, \text{ and } \int \psi_n - \int \phi_n \to 0$$

If ϕ_n and ψ_n are any sequences of step functions satisfying above, then as $n \to \infty$

$$\int \phi_n \to \int f \text{ and } \int \psi_n \to \int f$$

Lemma 1: Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function with bounded support $[a, b]$. The following are equivalent:

- f is Riemann-Integrable
- For every $\epsilon > 0$ there exists $a = x_0 < ... < x_n = b$ such that if M_j and m_j denote the supremum and infimimum values of f on $[x_{j-1}, x_j]$ respectively then

$$\sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \epsilon$$

- For every $\epsilon > 0$ there exists $a = x_0 < ... < x_n = b$ such that, with $I_j = (x_{j-1}, x_j)$ for $j \geq 1$:

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)| \, |I_j| < \epsilon$$

Theorem 3: Suppose f and g are R-I, and α, β are real numbers. Then:

- $\alpha f + \beta g$ is R-I and the integral is what you expect
- If $f \geq 0$ then $\int f \geq 0$, if $f \leq g$ then $\int f \leq \int g$
- $|f|$ is R-I and $|\int f| \leq \int |f|$
- $\max\{f, g\}$ and $\min\{f, g\}$ are R-I
- fg is R-I

Theorem: If f is not zero outside some bounded interval then it is not integrable.
Theorem 4: If $g : [a, b] \to \mathbb{R}$ is continuous, and f is defined by $f(x) = g(x)$ for $a \leq x \leq b$, $f(x) = 0$ for $x \notin [a, b]$, then f is R-I.
Fundamental Theorem of Calculus: Let $g : [a, b] \to \mathbb{R}$ be R-I. For $a \leq x \leq b$ define

$$G(x) = \int_a^x g$$

G is differentiable on (a, b) and its derivative is $g(x)$.
Theorem 5: Let $g : [a, b] \to \mathbb{R}$ be R-I. For $a \leq x \leq b$ let $G(x) = \int_a^x g$. Suppose g is continuous at x for some $x \in [a, b]$. Then G is differentiable at x and $G'(x) = g(x)$.
Theorem 6: Suppose $f : [a, b] \to \mathbb{R}$ has continuous derivative f' on $[a, b]$. Then

$$\int_a^b f' = f(b) - f(a)$$

Theorem 7: Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of R-I functions which converges uniformly to a function f. Suppose that f_n and f are zero outside some common interval $[a, b]$. Then f is R-I and

$$\int f = \lim_{n \to \infty} \int f_n$$

Corollary: Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of R-I functions such that $\sum_n f_n$ converges uniformly to a function f. Suppose that f_n and f are zero outside some common interval $[a, b]$. Then $f = \sum_n f_n$ is R-I and

$$\int \sum_n f_n = \sum_n \int f_n$$

Integral Test: Suppose $(a_n)_{n=1}^{\infty}$ is a non negative sequence of numbers and $f : [1, \infty) \to (0, \infty)$ is a function such that:

- $\int_1^n f \leq K$ for some K and all n
- $a_n \leq f(x)$ for $n \leq x < n + 1$

Then $\sum_n a_n$ converges to a real number which is at most K.
Proof: Take $\phi = \sum_{k=1}^{n} a_k \chi_{[k,k+2)}$ is a step function which satisfies $\phi \leq f \chi_{[1,n+1)}$ so that:

$$\sum_{k=1}^{n} a_k = \int \phi \leq \int f \chi_{[1,n+1)} = \int_1^{n+1} f \leq K$$

Integral Examples:

- If $f : \mathbb{R} \to \mathbb{R}$ is R-I, then f must be bounded and have bounded support.
  Proof: If f is R-I then taking $\epsilon = 1$, there exists step functions such that $\phi \leq f \leq \psi$. Then $|f| \leq \max\{|\phi, |\psi|\}$, which as a step function, takes only finitely many values, therefore is bounded. So f is bounded. Moreover, there are $M, N \in \mathbb{R}_+$ such that $\phi(x) = 0$ for $|x| > M$ and $\psi(x) = 0$ for $|x| > N$, so that $\phi(x) = \psi(x) = 0$ for $|x| > \max\{M, N\}$. Since $\phi \leq f \leq \psi$ this forces $f(x) = 0$ for $|x| > \max\{M, N\}$, and so f has bounded support.
- Not zero outside some bounded interval $\implies$ not integrable.
  Not bounded $\implies$ not integrable.
  Proof: Since every step function is bounded and vanishing outside a bounded interval, the fact that $\phi_n \leq f \leq \psi_n$ implies the same for f.
- $\sum_{n=1}^{N} r^n = \frac{r^{N+1} - 1}{r - 1}$ provided $-1 < r < 1$
- Converges only if: $p > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

- This function is R-I:

$$f(x) = \begin{cases} 1 & x = 1/n^2, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

  Proof: Takes infintely many different values so not a step function. Define $\phi = 0$ and $\psi(x) = 1$ for $0 \leq x \leq 1/N^2$ and $x = 1/n^2, 1 \leq n \leq N$ and $\psi(x) = 0$ otherwise. We get $\phi \leq f \leq \psi$, and $\int \phi = 0$, $\int \psi = 1/N^2$. So f integrable and $\int f = 0$.
- $\chi_{\mathbb{Q} \cap [0,1]}$ Is not R-I.
  Proof: let there be step functions such that $\phi \leq \chi_{\mathbb{Q} \cap [0,1]} \leq \psi$. Then on any interval of positive length on which ϕ is constant, the value of ϕ must in fact be non-positive. This is because any interval of positive length must contain irrationals, and we have that $\chi_{\mathbb{Q} \cap [0,1]}(x) = 0$ for irrational x. Thus $\phi(x) \leq 0$ except for possibly finitely many values of x, and therefore $\int \phi \leq 0$. Similarly any interval of positive length must contain rationals, ψ must be at least 1 on any interval of positive length meeting $[0, 1]$ on which it is constant. Therefore $\int \psi \geq 1$. Hence $\int \psi - \int \phi \geq 1$. So not true that $\forall \epsilon > 0$ there exist step functions ϕ and ψ such that $\phi \leq \chi_{\mathbb{Q} \cap [0,1]} \leq \psi$ and $\int \psi - \int \phi < \epsilon$.

Metric Spaces

Metric Space: is a set X with a function $d : X \times X \to \mathbb{R}$ which satisfies the following for all $x, y, z \in X$:

- Positive Definite: $d(x,y) \geq 0$ with $d(x,y) = 0$ iff $x = y$
- Symmetric: $d(x,y) = d(y,x)$
- Triangle Inequality: $d(x,y) \leq d(x,z) + d(z,y)$

The Usual Metric: Every Euclidean space $\mathbb{R}^n$ is a metric space with metric $d(x,y) = ||x-y||$.
The Discrete Metric: $\mathbb{R}$ is a metric space with metric

$$\sigma(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Examples:

- $d_1(x,y) = \sum_{i=1}^n |x_i - y_i|$ (looks like unit diamond
- $d_2(x,y) = ||x-y|| = (\sum_{i=1}^n |x_i - y_i|^n)^{\frac{1}{n}}$ (looks like the unit circle)
- $d_\infty(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|$ (looks like the unit square)

Function Metric Spaces: on $C([0,1])$ we have:

- $$d_1(f,g) := \int_0^1 |f-g|$$
- $$d_2(f,g) := \left(\int_0^1 |f-g|^2\right)^{1/2}$$
- $$d_\infty(f,g) = \sup_x |f(x) - g(x)|$$

Strongly Equivalent: if between two metrics d and ρ on a set X, there exists positive numbers A and B such that:

$$d(x,y) \leq A\rho(x,y) \text{ and } \rho(x,y) \leq Bd(x,y) \text{ for all } x, y \in X$$

Equivalent: if between two metrics d and ρ on a set X, for every $x \in X$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$d(x,y) < \delta \implies \rho(x,y) < \epsilon$$

and

$$\rho(x,y) < \delta \implies d(x,y) < \epsilon$$

Completeness and Contraction

Converging: (x_n) converges in X if there is a point $a \in X$ such that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that:

$$n \geq N \implies d(x_n, a) < \epsilon$$

Cauchy: (x_n) is Cauchy if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that:

$$n, m \geq N \implies d(x_n, x_m) < \epsilon$$

Complete Metric Space: If every Cauchy sequence of points in the metric space has a limit that is also in the metric space. Or that every Cauchy sequence in M converges in M.
Contraction: Let (X,d) be a metric space. A function $f : X \to X$ is called a contraction if there exists a number α with $0 < \alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x,y) \text{ for all } x, y \in X$$

Banach's Contraction Mapping Theorem: If (X,d) is a complete metric space and if $f : X \to X$ is a contraction, then there is a unique point $x \in X$ such that $f(x) = x$. This point x is called a fixed point of f.
Proof: Pick $x_0 \in X$, and let $x_1 = f(x_0), x_2 = f(x_1), ..., x_{n+1} = f(x_n)$. Consider $d\left(x_{n+1}, x_n\right) = d\left(f\left(x_n\right), f\left(x_{n-1}\right) \leq \alpha d\left(x_n, x_{n-1}\right)\right)$. Repeating we get $d\left(x_{n+1}, x_n\right) \leq \alpha^n d\left(x_1, x_0\right)$.
So that when $m \geq n$

$$d\left(x_m, x_n\right) \leq d\left(x_m, x_{m-1}\right) + \ldots + d\left(x_{n+1}, x_n\right)$$

$$\leq \left(\alpha^{m-1} + \ldots + \alpha^n\right) d\left(x_1, x_0\right)$$

$$\leq \frac{\alpha^n}{1-\alpha} d\left(x_1, x_0\right)$$

Since $\alpha < 1$. This shows that (x_n) is a Cauchy sequence in X, and since X is complete, there exists $x \in X$ to which it converges. Now a contraction map is continuous, so continuity of f at x shows that $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$, so indeed $f(x) = x$. Finally, if there are $x, y \in X$ with $f(x) = x$ and $f(y) = y$, we have $d(x,y) = d(f(x), f(y)) \leq \alpha d(x,y)$. Which since $\alpha < 1$, forces $d(x,y) = 0 \implies x = y$.

Compactness in Metric Spaces

Cover: A covering of X is a collection of sets whose union is X. An open covering of C is a collection of open sets whose union is X.
Compact: For Q to be compact: for every open cover of Q there is a finite subcover.
Negation: There exists some open cover $\{U_\alpha\}$ of Q which has no finite subcover.
More in Depth Compactness Definition: For every collection of open sets $\{U_\alpha\}$ in $\mathbb{R}^2$ such that $Q \subseteq \cup_\alpha U_\alpha$ there is a finite subcollection $\{U_{\alpha_1}, ..., U_{\alpha_k}\}$ such that $Q \subseteq \cup_{j=1}^k U_{\alpha_j}$.
Proposition: A subset $A \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.
Corollary: A compact set is always closed. A closed subset of a compact set is compact.
Theorem: Suppose (X,d) is a complete metric space, which for all $r > 0$ we can cover X by finitely many closed balls of radius $r > 0$. Then X is compact.
Proof Sort Of: Suppose X is compact. Consider the open cover given by $\{B(x,r) : x \in X\}$. This has a finite subcover $\{B(x_j, r) : 1 \leq j \leq n\}$. Then the closed balls of radius r with centres at x_j cover X.
Lemma: If X is compact then it is sequentially compact. i.e every sequence in X has a convergent subsequence.
Compact Functions: The direct image of a compact metric space by a continuous function is compact. i.e let X and Y be metric spaces, with X compact, and let $f : X \to Y$ be a continuous surjective map. Then Y is compact.
Continuity: For a mapping $f : X \to Y$, it is continuous if:

$$(\forall \epsilon > 0)(\forall x \in X)(\exists \delta > 0)\left(\forall x' \in X\right)$$
$$d_X\left(x, x'\right) < \delta \Longrightarrow d_Y\left(f(x), f\left(x'\right)\right) < \epsilon$$

Uniform Continuity: For a mapping $f : X \to Y$, it is uniformly continuous if:

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in X)\left(\forall x' \in X\right)$$
$$d_X\left(x, x'\right) < \delta \Longrightarrow d_Y\left(f(x), f\left(x'\right)\right) < \epsilon$$

Proposition: Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous.
Cluster Point: $a \in X$ is a cluster point iff $B_\delta(a)$ contains infinitely many points for each $\delta > 0$
Bolzano-Weierstrass Property: X satisfies the Bolzano-Weierstrass Property iff every bounded sequence $x_n \in X$ has a convergent subsequence.

Heine-Borel: Let X be a separable metric space which satisfies the Bolzano-Weierstrass Property, and H be a subset of X. Then H is compact iff it is closed and bounded.

Metric Examples:

- Proof of Cauchy-Shwarz $\int fg \leq \left(\int f^2\right)^{1/2}\left(\int g^2\right)^{1/2}$. Just expand $\int(f-\alpha g)^2 \geq 0$ and find the determinant when it has at most one real root.
- On the complete metric space $C([0,1])$ with the metric $d_\infty(f,g) = \sup)t \in [0,1]|f(t)-g(t)|$. Consider the mapping $T : C([0,1]) \to C([0,1])$ given by
$$T(f)(s) = \int_0^s f(t)(s-t)\mathrm{d}t + g(s)$$
Nor for any $0 \leq s \leq 1$ we have:
$$T(f)(s) - T(h)(s) = \int_0^s (f(t)-h(t))(s-t)\mathrm{d}t$$
$$|T(f)(s) - T(h)(s)| \leq \int_0^s |f(t)-h(t)|(s-t)\mathrm{d}t$$
$$\int_0^s |f(t)-g(t)|(s-t)dt \leq \sup_{0\leq t\leq 1} |f(t)-g(t)| \int_0^s (s-t)dt$$
$$= \frac{s^2}{2} d_\infty(f,g) \implies d_\infty(Tf,Th) \leq \frac{1}{2} d_\infty(f,h)$$
Since $\alpha < 1$ T is a contraction $\implies \exists$ unique fixed point $f \in C([0,1])$ of T.
- Function metrics d_1 and d_∞ aren't strongly equivalent. Proof: $f_n(x) = x^n$, $d_1(f_n,0) = 1/(n+1)$ and $d_\infty(f_n,0) = 1$ for all n.
- Show that $f(x) = 2 + x^{-2}$ on $[2,\infty)$ is a contraction mapping $[2,\infty)$ into itself.
Proof: For $x \geq 0$ we have $f(x) \geq 2$ and hence the map maps $[2,\infty)$ into itself. We check that f is a contraction. Clearly:
$$|f(x)-f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|f'(c)\right| |x-y|$$
for some $c \in [2,\infty)$ by MVT. Since $|f'(c)| = \frac{2}{|c^3|} \leq \frac{1}{4} < 1$ the map is a contraction.
- If d and ρ are strongly equivalent, then they are equivalent.
Proof: If $\epsilon > 0$ then choosing $\delta = \epsilon/B$ works for the first statement, and $\delta = \epsilon/A$ works for the second statement for all x. So $\delta = \min\{\epsilon/A, \epsilon/B\}$ works for both.
- Let (X,d) be a discrete metric space, then it is complete. Proof: Suppose (x_n) is Cauchy in (X,d). Take $\epsilon = 1$. Then $\exists N$ such that for $m,n \geq N$ we have $d(x_m,x_n) < 1$. But this means for $m,n \geq N$ we have that $x_m = x_n$, i.e that x_n is constant for $n \geq N$. Hence (x_n) converges and so (X,d) is complete.

Spaces:

Interior: Let E be a subset of a metric space X. The interior of E is:
$$E^o := \bigcup\{V : V \subseteq E \text{ and } V \text{ is open in } X\}$$
Closure: Let E be a subset of a metric space X. The closure of E is:
$$\overline{E} := \bigcap\{B : B \supseteq E \text{ and } B \text{ is closed in } X\}$$
$$\overline{E} = \{x \in X : x \text{ is a limit point of E}\}$$
Theorem:

- $E^o \subseteq E \subseteq \overline{E}$
- If V is open and $V \subseteq E$, then $V \subset E^o$
- If C is closed and $E \subseteq C$, the $\overline{E} \subseteq C$

Boundary: let $E \subseteq X$, then the boundary of E is:
$$\partial E := \{x \in X : \forall r > 0, B_r(x) \cap E \neq \emptyset \text{ and } B_r(x) \cap E^c \neq \emptyset\}$$
Theorem: Let $E \subseteq X$. Then $\partial E = \overline{E} \backslash E^o$
Separable: Metric space X is separable iff it contains a countable dense subset. i.e iff there is a countable set Z of X such that for every point $a \in X$ there is a sequence $x_k \in Z$ such that $x_k \to a$ as $k \to \infty$
Relatively Open: A set $U \subseteq E$ is relatively open in E iff there is a set V open in X such that $U = E \cap V$
Relatively Closed: A set $U \subseteq E$ is relatively closed in E iff there is a set C closed in X such that $A = E \cap C$

Spaces Examples:

- $\mathbb{R}$ is closed and open. $\text{int}\mathbb{R} = \mathbb{R}$, $\overline{\mathbb{R}} = \mathbb{R}$, $\partial\mathbb{R} = \emptyset$. Not compact, but connected since path connected.
- $\mathbb{Q}$ is not open, not closed. $\text{int}\mathbb{Q} = \emptyset$, $\overline{\mathbb{Q}} = \mathbb{R}$, $\partial\mathbb{Q} = \mathbb{R}$. Not compact, not connected.
- $A \subset \mathbb{R}$ is connected iff A is any type of interval (open, closed or semi-open).
- $E = \bigcup_{n=1}^\infty \left\{\frac{1}{n}\right\} \times [0,1]$ Is neither open nor closed. $\text{int}E = \emptyset$, $\partial E = \overline{E} = E \cup \{0\} \times [0,1]$. Since it is not closed it is not compact. It is not connected, we can take $U = \{(x,y) : x < 3/4\}$ and $V = \{(x,y) : x > 3/4\}$. These are two open disjoint sets that separate E.

Definitions

- **Injective:** $f(a) = f(b) \implies a = b$
- **Surjective:** $\forall y \in Y, \exists x \in X$ such that $y = f(x)$
- **Closed:** A set $F \subset X$ is closed iff the complement $X \backslash F$ is open. That is for any $x \in X \backslash F$ there is $r > 0$ such that $B(x,r) \subset X \backslash F$.
- **Open Ball:** Let $a \in X$ and $r > 0$. Then the open ball with centre a and radius r is the set:
$$B(a,r) := \{x \in X : d(x,a) < r\}$$
- **Closed Ball:** Let $a \in X$ and $r > 0$. Then the closed ball with centre a and radius r is the set:
$$B(a,r) := \{x \in X : d(x,a) \leq r\}$$
- **Support:** The support of a real valued function is a subset of the domain containing elements which are not mapped to zero. If the domain is a topological space, the support is instead the smallest closed set containing all points not mapped to zero.
- A countable union of countable sets is countable.
- **Compact (set):** Closed and bounded
- **Connected (set):** If path connected. i.e connected space is a topological space that cannot be represented as the union of two or more disjoint non-empty open subsets. i.e
A subset $A \subset X$ is connected if there does not exist open and disjoint sets $U, V \subset X$ such that
$$A \cap U \neq \emptyset, B \cap V \neq \emptyset, \text{ and } A \subset U \cup V$$
- **Radius of Convergence:** The radius of convergence R of the given power series is the unique number R such that the series converges for $|x| < R$ and diverges for $|x| > R$. We have $R \in [0,\infty) \cup \{\infty\}$ where when $R = 0$ the series only converges at $x = 0$ while $R = \infty$ means that the power series converges for all $x \in \mathbb{R}$.