How to be Good at Math

1. A note to the reader.

This book is for anyone currently frustrated by math. Whatever level you are at in your studies, there is something in these pages you may find helpful. The reading level varies by chapter, but is most challenging in the preface and introduction. If you find this note difficult to read then it may be advisable to skip the first two sections and jump straight to chapter one. My primary source for this book is my own experience as a student and tutor. Math education may be relevant to my vocational and educational background, but I claim no special authority on the subject endowed by any institution. I am not a schoolteacher, professor, scholar of math pedagogy, or even a college graduate. I have not stayed fully up to date with every trend or development in contemporary cirriculum. Furthermore, when writing this book I made no efforts to avoid overgeneralizations, or to stay on message. Reactionary philosophical tirades should be expected. Readers who already have a background in the theory of math education or the philosophy of mathematics will probably think I am beating a dead horse. I would respond that although this horse has been being vicously beaten for decades, it is not yet dead. In other words, none of my arguments are new, but they remain relevant. The critical reader will find many statements herein which are strictly false, unnecesarily provocative, and perhaps even offensive. What I can say is that every statement--true or false--was included for a reason: be it stylistic or rhetorical.

2. Preface: Some hard truths for those who want to love math.

Some will tell you that the way to become good at math is to learn to "love" math. This is well intentioned advice. It even approaches the truth of the matter. Taken on face value however, it is an endorsement of self-delusion. Math sucks. It doesn't just suck a little bit, it sucks a lot. Realisticly, forcing math upon unwilling participants is inhumane. Math is always either so easy that it's boring, or so hard that it's infuriating. Some are inclined to enjoy math for this very reason. There is a certain strain among mathematicians which can only be described as masochistic, or at least obsessed with one of the following: pleasing authority, proving their intelligence, or just intellectual challenge for it's own sake. I doubt these traits would provide sufficient motivation to pursue mathematics on a long term basis, but they may very well get some mathematicians to the point where they catch a glimse of the sublime through their work. This is the point of no return. Those of us who have passed through it are beyond salvation. We may take prolonged breaks from math, but it will always draw us back. Don't be fooled however. Math may be beautiful, it may be ludicrously useful, it may sometimes produce states of mind that can only be described as transcendent, and on occaision it may even be fun, but it is certainly not good. Above all else, math demands discipline, labor, and suffering. The only sane response to math is to denounce it as what it is: a Pandoras Box. Not only does it inflict suffering onto billions of students worldwide, but it is also an existential threat to humanity. Whether your vision of the apocolypse is one of artificial intelligence run wild, or of the more immediate threat of nuclear annihilation, there is but one common culprit: math.

For a small portion of the population, math may never be particularly difficult. If this describes you, then you may not find anything in this book you can relate to. These sorts of people seem to "just get math". I don't get these people. Their tendency to act as bewlidered by their ability as I am has always seemed downright conspiratorial to me. I have often wondered if there is a second version of every textbook given to a small percentage of each math class class which explains how the subject actually works, and the rest of us where being asked to memorize fragments chosen intentionally to seem arbitrary and induce frustration. I've had friends in this category tell me they make a habit of checking their test problems a half dozen times, not out of fear of missing a problem, but out of fear that their classmates would resent them for getting to leave a 2 hour exam after the first 20 minutes. I wish I could say this fear is unrealistic.

Even if you never noticed how much math sucks, you have problably noticed the social pitfalls that comes with being a "math person". Saying "I love math" usually illicits the response I'd expect to see after saying "My favorite book is The Diary of Anne Frank", or asking "So how old where you when your first childhood pet died?" Indeed, bringing up math in casual conversation is frequently met with that thousand yard stare you see after asking someone to recall a traumatic memory. So why is math class a traumatic experience for so many people? A portion of the blame can be attributed to the way math is taught in public schools. Too often, math is reduced to a set of operations and formulas that students are expected to memorize. In these settings, testing does not measure conceptual understanding, but rather computational proficiency--a skill which in most situations is rendered practically obsolete by modern day computer technology. I sometimes disparigingly refer to this form of mathematics as "formula-0 racing", because like formula one racing it priveleges speed above all else, but unlike formula-1 racing, it is pointless. The main problem with this approach is that it encourages students to see their mathematical ability as correlated with how fast they can work through problems. I wouldn't discourage anyone from developing computational proficeny, especially if they intend to pursue math in an academic setting, but it is not a precondition--nor a replacement--for conceptual understanding.

To see what I mean by formula-0 racing, let us take an example. A test on multiplication might present the student with a dozen problems, each simply asking for the product of two numbers. The answers a student provides might look a number of ways. One student might only provide correct answers to the most familiar problems (10x10, 5x5 and so on). This would indicate a lack of understanding of multiplication as an operation, but that the student has memorized the answers to certain problems. A second student might provide to correct answer to every problem, indicating that they understand multiplication well enough to find the product of two numbers, and that they where able to finish fast enough to go back and check their work. Another student might answer most problems correctly, but either get several wrong or not complete all problems. This would indicate the same degree of understanding as the first student, but that the time constraints of the test posed an issue to them. This student may be able to improve their performance on the test by spending more time practicing amultiplication problems, but does this do anything to improve their understanding of multiplication? No! Multiplication is an operation which applies to infinite sets. No one will ever memorize every multiplication problem, and furthermore there is no way to increase the proportion of multiplication problems that you know the answer to. Being asked to "learn your times tables" is like Sysaphus being asked to roll his boulder up a mountain. The most valuable thing one can learn from a task like this, is precisely why it is pointless. Testing students in this way implies a disregard for the value of their time, and teaches them to confuse mathematical understanding with route memorization.

Let us compare this formula-racing test to another hypothetical test. Instead of a dozen problems, this one only has two. The first is a multiplication probem which the students are unlikely to have encountered previously. Answers to this problem alone tells us everything that we could learn about a students understanding of multiplication from the 12 problems of the previous test. The second question simply asks the student to define multiplication. Answers to this question may vary widely and will paint a much clearer picture of students understanding of the operation. One student might write that multiplication is the little x that goes in between two numbers. This tells us that they recognize the symbol for multiplication, but that they are confused about what exactly it is. Another student might define multiplication in terms of the area of a square with side lengths equal to the factors. This would indicate both a practical understanding of multiplication, as well as that the student is drawing connections to geometry, but that they are thinking of multiplication in concrete rather than abstract terms. Another student might provide a definiton using variables and sets, and also provide a list of the algebraic properties of multiplication. This would indicate a stronger conceptual and practical understanding of the operation than any of the previous answers, but if we had used our formula-racing test we would not have any way of recognizing this.

Math's hardness problem is not merely a result of pedagogical convention. As a field of study, math is unique in it's logical rigour and propensity to constantly introduce new symbols and definitions. I wouldn't even call it uniquely backwards in how it is taught, but the information which is left out of most mathematics cirriculum, in conjunction with math's inherent difficulty, does create a uniquely stressful environment for many if not most students. This booklet is meant to provide students--at any level--with a set of strategies that will assist in develop a conceptual and practical understanding of whatever area of math they are studying. Some of what I endorse here is already common practice in math classes taken by graduate students and upperclassmen in college, and has historical precident in every level of the field. If you aren't interested in math as a field of study or a skill, but instead just want to pass a class, then just see the last chapter: How to Cheat at Math.

3. Introduction: What is mathematics?

Ask math students at various levels to define math, and you are likely to get a variety of answers. If you are studying arithmetic, you might see math as anything involving numbers (1, 2 ,7). This is not too different from the Aristotelian definition: "mathematics is the science of quantity". Perhaps you've put more thought into it and realized that math also involves a class known as operators (+, -, %). Maybe you've gotten to elementary algebra and you know math also involves variables, as well as some mysterious thing called a "function". By calculus you are probably confused enough to not even try and come up with a definition, but these function things seem important. By linear or abstract algebra you may have contracted some funny, Russelesque ideas about the all encompassing power of sets. You continue refining this deinition until arriving at one of the countless variations on what seems to be the most formal popular definition of math: "mathematics is the study of logic applied to quantity, arrangement, and other properties of--or relations between--sets". A more succint, abstract, and formalist definition would be that of Walter Worwick Steyer: "mathematics is the study and classification of all possible patterns". This is the best definition of math that I've come across so far, but I beleive it leaves out something important. What of mathematics as a mode of interpersonal communication and creative expression? Would we call music "the study and classification of notes and chords"? This definition of music should strike most people as reductive, leaving out music's most essential qualities. So why shouldn't we say the same for mathematics?

Perhaps it sounds heady to call math an art form, but I don't think it is unjustified. All forms of art--writing, drawing, painting, photography, or music--are processes by which we create representations of what would otherwise be in some way innacessible to our audience. In the case of photography, this is usually thing in the world, as seen from a specific setting. Some realist paintings are similar to photography in this way, but due to the unique position of painting in institutional art, painting often serves to make theoretical or political statements about the art itself. Writing has an extremely broad scope, but the nature of words means it lacks the capacity for specificty seen in paintings or photographs. It would be quite a literary feat to tell you in words what I saw as I watched such a man leave such a resturant on such a day in any detail, but visual mediums can do this easily. Music is even more broad in the sense that there is a massive quantity of building blocks to work with, but since each note is devoid of meaning on it's own, music is somewhat limited to making statements that are abstract and often non-representational. Why this gives music such a uniquely powerful capacity to communicate internal emotions is a question for another day, but I would encourage the interested reader to look into Arthur Schopenhauer's takes on the matter.

Mathematics is no different from all these art forms in that it involves using a set of procedures and physical tools to communicate. Precisely what is being communicated may be somewhat elusive, but the procedures and tools themselves are not. Just as painting uses brushes, paint, and a canvas, math uses pens, paper, and sometimes a computer. As painting uses lines, fields, colors and textures, math uses numbers and symbols. "But what about all the rules?" I hear you object "Art doesn't have rules that you always need to follow." There is a difference in proportion here but ultimately, math doesn't either. Like all art, math can be used to present false or misleading information. This might make it bad or incorrect, but it doesn't make it "not math" just like incorrect written statements aren't "not writing". All forms of art have rules, and all forms of artistic innovation involve breaking these rules. Mathematical innovation is no different. Breaking the rules in a systematic and rigorous way is exactly how you can begin to make mathematical discoveries, just like breaking conventions of baroche music drove the development of later forms of classical music. The definition of math which I will be using from here on is as follows: "mathematics is the use of symbols to depict abstract relations." This definition excludes many forms of applied mathematics, which may rub some the wrong way. You can remedy this by ommiting the word "abstract", but depending on how you define symbols you may be including those hearts with initials inside that you see carved into trees.

4. Section 1: Use your words.

I'd wager that you don't think in terms of mathematical symbols. I'd also wager that at least some of the sheets of paper you do math on have very few words written on them. Symbols are essential to math, but only using symbols is not an effective way to present information. The first step towards understanding any concept in math, is putting that concept into words. Words are how we think, and are the primary way we communciate. Annotating my work in plain language is single best way I have found to be sure I am always drawing new connections and identifing points of confusion. Next time you are working through a problem, try writing a one or two sentence justification for each step in the process. Cite relevant properties or identites. This will slow you down a bit at first, but it will be worth it. Your teachers will be better able to give you guidance on areas you are struggling with becaus they will be able to see what you where thinking when you distributed that coefficent to only one term inside the parentheses, or treated that function as though it had algebraic properties it lacks. You will also find yourself catching far more mistakes on your own. If you can't think of anything to write then just write a joke, it will keep you in the habit of annotating your work and maybe whoever reads it will chuckle.

Let us take examples of solutions to problems from various areas of mathematics and demonstrate what I mean. The first 7 problems have been taken from the popular educational math website: Khan Academy. The url to this site is khanacademy.org. It is a fantastic recourse for practicing math common to American shool cirriculum.

We begin with pre-algebra. A word problem about ratios reads "Ken can walk 40 dogs in 8 hours. How many dogs can he walk in 12 hours?" Figure 1.1 shows a potential student's solution which includes few words and no sentences. This style of showing your work is fairly typical among math students at a pre-college level. Notice how there is a mistake made in the second line. How easy is it for you to identify where the mistake was made? Perhaps not so difficult, but imagine that you are a math teacher for middle school students and you have to grade hundreds of problems like this. Would you take the time to pinpoint exactly where the mistake was made?

[Insert Figures 1.1 and 1.2 side by side]

Now compare figure 1.1 to figure 1.2. This imaginary student made all the same steps to solve the problem, and got the same, incorrect answer. However, this student took the time to annotate their work, explaining their rationale for each step. How easily can you identify the error? If you where a teacher grading these two solutions--although they both ultimately provide the same answer--which student would you think had a better understanding of the content? When it comes time to look back on this problem to pinpoint areas of improvement, which student do you think will have an easier time finding the mistake they made?

Now, let's take an example from the first half of elementary algebra. This problem provides an ineqality with expressions on both sides and asks us to solve for y. The inequality itself is written at the top of both figures 1.3 and 1.4.

[insert figures 1.3 and 1.4 side by side]

Unlike our first example, the first figure here shows a correct answer, and the second shows and incorrect answer. The work for each is nearly identical though, and the first student seems to have almost made the same mistake. Take a look at the writing in red. This was written by our imaginary math teacher when she was grading their work. Note how in figure 1.3, the only thing our teacher was able to comment on was the presentation of the students work. In figure 1.4 however, the annotations provide our teacher with enough information to point out what may be a conceptual misunderstanding: the students confusion about whether what they are solving is an inequality or an equation. The word choice may have been a meer oversight, but if it isn't, this could explain why the student didn't think to remember that the sign needed to be flipped later on.

Our third example uses a problem from pre-algebra.

Furthermore, try and develop an aesthetic appriceation for mathematics by seeking out concepts and proofs that you find beautiful. The rest of this chapter is comprised of a collection of my personal favorites.

5. Section 3: Consider the set.

6. Section 4: Narrative is key.